National Finals 2019 – Challenger Division

April 28 — May 4, 2019

Time Limit: 2 hours.
Each problem is worth 1 point.

1. At a math competition, a team of 8 students has 2 hours to solve 30 problems. If each problem needs to be solved by 2 students, on average how many minutes can a student spend on a problem?

2. A *trifecta* is an ordered triple of positive integers (a, b, c) with $a < b < c$ such that a divides b, b divides c, and c divides ab. What is the largest possible sum $a + b + c$ over all trifectas of three-digit integers?

3. Determine all real values of x for which
$$\frac{1}{\sqrt{x}+\sqrt{x-2}}+\frac{1}{\sqrt{x}+\sqrt{x+2}}=\frac{1}{4}.$$

4. How many six-letter words formed from the letters of AMC do not contain the substring AMC? (For example, AMAMMC has this property, but AAMCCC does not.)

5. What is the largest integer with distinct digits such that no two of its digits sum to a perfect square?

6. Seven two-digit integers form a strictly increasing arithmetic sequence. If the first and last terms of this sequence have the same set of digits, what is the sum of all possible medians of the sequence?

7. Triangle ABC has $AB = 8, AC = 12, BC = 10$. Let D be the intersection of the angle bisector of angle A with BC. Let M be the midpoint of BC. The line parallel to AC passing through M intersects AB at N. The line parallel to AB passing through D intersects AC at P. MN and DP intersect at E. Find the area of $ANEP$.

8. The Fibonacci sequence $F_0, F_1, \ldots$ satisfies $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Compute the number of triples (a, b, c) with $0 \leq a < b < c \leq 100$ for which F_a, F_b, F_c is an arithmetic progression.

9. How many decreasing sequences $a_1, a_2, \ldots, a_{2019}$ of positive integers are there such that $a_1 \leq 2019^2$ and $a_n + n$ is even for each $1 \leq n \leq 2019$?

10. Let a, b be positive real numbers with $a > b$. Compute the minimum possible value of the expression
$$\frac{a^2b - ab^2 + 8}{ab - b^2}.$$

11. Let ABC be a right triangle with hypotenuse AB. Point E is on AB with $AE = 10BE$, and point D is outside triangle ABC such that $DC = DB$ and $\angle CDA = \angle BDE$. Let $[ABC]$ and $[BCD]$ denote the areas of triangles ABC and BCD. Determine the value of $\frac{[BCD]}{[ABC]}$.

12. Determine the number of 10-letter strings consisting of As, Bs, and Cs such that there is no B between any two As.

13. The infinite sequence $a_0, a_1, \ldots$ is given by $a_1 = \frac{1}{2}$, $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine the infinite product $a_1a_2a_3\cdots$.

14. In a circle of radius 10, three congruent chords bound an equilateral triangle with side length 8. The endpoints of these chords form a convex hexagon. Compute the area of this hexagon.

15. Let $P(x)$ be a polynomial with integer coefficients such that

$$P(\sqrt{2}\sin x) = -P(\sqrt{2}\cos x)$$

for all real numbers x. What is the largest prime that must divide $P(2019)$?

16. What is the product of the factors of 30^{12} that are congruent to 1 modulo 7?

17. Tommy takes a 25-question true-false test. He answers each question correctly with independent probability $\frac{1}{2}$. Tommy earns bonus points for correct streaks: the first question in a streak is worth 1 point, the second question is worth 2 points, and so on. For instance, the sequence TFFTTTFT is worth $1 + 1 + 2 + 3 + 1 = 8$ points. Compute the expected value of Tommy's score.

18. Two circles with radii 3 and 4 are externally tangent at P. Let $A \neq P$ be on the first circle and $B \neq P$ be on the second circle, and let the tangents at A and B to the respective circles intersect at Q. Given that $QA = QB$ and AB bisects PQ, compute the area of QAB.

19. Let n be the largest integer such that 5^n divides $12^{2015} + 13^{2015}$. Compute the remainder when $\frac{12^{2015}+13^{2015}}{5^n}$ is divided by 1000.

20. Kelvin the Frog lives in the 2-D plane. Each day, he picks a uniformly random direction (i.e. a uniformly random bearing $\theta \in [0, 2\pi]$) and jumps a mile in that direction. Let D be the number of miles Kelvin is away from his starting point after ten days. Determine the expected value of D^4.

21. Let $ABCD$ be a rectangle satisfying $AB = CD = 24$, and let E and G be points on the extension of BA past A and the extension of CD past D respectively such that $AE = 1$ and $DG = 3$.

    Suppose that there exists a unique pair of points (F, H) on lines BC and DA respectively such that H is the orthocenter of $\triangle EFG$. Find the sum of all possible values of BC.

22. Find the largest real number λ such that

$$a_1^2+\cdots+a_{2019}^2 \geq a_1a_2+a_2a_3+\cdots+a_{1008}a_{1009}+\lambda a_{1009}a_{1010}+\lambda a_{1010}a_{1011}+a_{1011}a_{1012}+\cdots+a_{2018}a_{2019}$$

    for all real numbers $a_1, \ldots, a_{2019}$. The coefficients on the right-hand side are 1 for all terms except $a_{1009}a_{1010}$ and $a_{1010}a_{1011}$, which have coefficient λ.

23. For Kelvin the Frog's birthday, Alex the Kat gives him one brick weighing x pounds, two bricks weighing y pounds, and three bricks weighing z pounds, where x, y, z are positive integers of Kelvin the Frog's choice.

    Kelvin the Frog has a balance scale. By placing some combination of bricks on the scale (possibly on both sides), he wants to be able to balance any item of weight $1, 2, \ldots, N$ pounds. What is the largest N for which Kelvin the Frog can succeed?

24. Let ABC be a triangle with $\angle A = 60°$, $AB = 12$, $AC = 14$. Point D is on BC such that $\angle BAD = \angle CAD$. Extend AD to meet the circumcircle at M. The circumcircle of BDM intersects AB at $K \neq B$, and line KM intersects the circumcircle of CDM at $L \neq M$. Find $\frac{KM}{LM}$.

25. Determine the remainder when

$$\prod_{i=1}^{2016}(i^4+5)$$

    is divided by 2017.

26. The permutations of $OLYMPIAD$ are arranged in lexicographical order, with $ADILMOPY$ being arrangement 1 and its reverse being arrangement 40320. Yu Semo and Yu Sejmo both choose a uniformly random arrangement. The immature Yu Sejmo exclaims, "My fourth letter is L!" while Yu Semo remains silent. Given this information, let E_1 be the expected arrangement number of Yu Semo and E_2 be the expected arrangement number of Yu Sejmo. Compute $E_2 - E_1$.

27. For an integer n, define $f(n)$ to be the greatest integer k such that 2^k divides $\binom{n}{m}$ for some $0 \leq m \leq n$. Compute $f(1) + f(2) + \cdots + f(2048)$.

28. Alex the Kat plays the following game. First, he writes the number 27000 on a blackboard. Each minute, he erases the number on the blackboard and replaces it with a number chosen uniformly randomly from its positive divisors, including itself. Find the probability that, after 2019 minutes, the number on the blackboard is 1.

29. Let n be a positive integer, and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers. Alex the Kat writes down the n^2 numbers of the form $\min(a_i, a_j)$, and Kelvin the Frog writes down the n^2 numbers of the form $\max(b_i, b_j)$.

    Let x_n be the largest possible size of the set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, such that Alex the Kat and Kelvin the Frog write down the same collection of numbers. Determine the number of distinct integers in the sequence $x_1, x_2, \ldots, x_{10{,}000}$.

30. Let ABC be a triangle with $BC = a$, $CA = b$, and $AB = c$. The A-excircle is tangent to $\overline{BC}$ at A_1; points B_1 and C_1 are similarly defined.

    Determine the number of ways to select positive integers a, b, c such that

    - the numbers $-a + b + c$, $a - b + c$, and $a + b - c$ are even integers at most 100, and
    - the circle through the midpoints of $\overline{AA_1}$, $\overline{BB_1}$, and $\overline{CC_1}$ is tangent to the incircle of $\triangle ABC$.