National Finals 2019 – Challenger Division

Official Solution Key

April 28 — May 4, 2019

Time Limit: 2 hours.
Each problem is worth 1 point.

1. At a math competition, a team of 8 students has 2 hours to solve 30 problems. If each problem needs to be solved by 2 students, on average how many minutes can a student spend on a problem?

   Proposed by: Jeffery Yu.

   Answer: $\boxed{16}$.

   There are a total of $2 \cdot 30 = 60$ solves distributed over 8 students, so each student solves $\frac{60}{8} = \frac{15}{2}$ problems on average. Over 120 minutes, this averages to $\frac{120}{15/2} = 16$ minutes per problem.

2. A *trifecta* is an ordered triple of positive integers (a, b, c) with $a < b < c$ such that a divides b, b divides c, and c divides ab. What is the largest possible sum $a + b + c$ over all trifectas of three-digit integers?

   Proposed by: Kevin Ren.

   Answer: $\boxed{1736}$.

   The constraints $a \mid b, b \mid c$ imply $a \leq \frac{1}{2}b$, $b \leq \frac{1}{2}c$. So, heuristically we would like $(a, b, c) = (x, 2x, 4x)$ where x is as large as possible. This requires $4x \mid 2x^2$, so x is even. The largest such solution is $(248, 496, 992)$, for a sum of 1736.

   Let us prove this is in fact the maximal sum $a + b + c$ over all trifectas (a, b, c). If $c \leq 992$, the bounds $a \leq \frac{1}{2}b, b \leq \frac{1}{2}c$ imply $(248, 496, 992)$ is optimal. If $992 < c < 1000$, then we cannot have $a = \frac{1}{2}b$ and $b = \frac{1}{2}c$ since, as we showed above, this requires a to be even and thus c to be a multiple of 8. So, in this case $a \leq \frac{1}{2} \cdot \frac{1}{3}c = \frac{1}{6}c$, and

   $$a + b + c \leq \left\lfloor \frac{999}{6} \right\rfloor + \left\lfloor \frac{999}{2} \right\rfloor + 999 = 1664 < 1736.$$

   Therefore 1736 is the maximal sum.

3. Determine all real values of x for which

   $$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x} + \sqrt{x+2}} = \frac{1}{4}.$$

   Proposed by: Alexander Katz.

   Answer: $\boxed{\frac{257}{16}}$.

   Rationalizing the denominator, we have that

   $$\frac{\sqrt{x} - \sqrt{x-2}}{2} + \frac{\sqrt{x+2} - \sqrt{x}}{2} = \frac{1}{4}.$$

   Thus $\sqrt{x+2} - \sqrt{x-2} = \frac{1}{2}$. Rearranging yields

   $$x + 2 = \left(\sqrt{x-2} + \frac{1}{2}\right)^2 = (x-2) + \sqrt{x-2} + \frac{1}{4}.$$

   Thus $\sqrt{x-2} = \frac{15}{4}$, and $x = \frac{257}{16}$.

4. How many six-letter words formed from the letters of AMC do not contain the substring AMC? (For example, AMAMMC has this property, but AAMCCC does not.)

   Proposed by: Kevin Ren.

   Answer: $\boxed{622}$.

   We use inclusion-exclusion. There are 3^6 six-letter words that can be formed from the letters of AMC. Of these, there are 3^3 each with AMC in positions 1-3, 2-4, 3-5, and 4-6, and one with AMC in two of these positions (AMCAMC). This produces a count of $3^6 - 4 \cdot 3^3 + 1 = 622$.

5. What is the largest integer with distinct digits such that no two of its digits sum to a perfect square?

   Proposed by: Kevin Ren.

   Answer: $\boxed{98652}$.

   Observe that no two of $a, 9-a$ can be digits, providing an immediate upper bound of 5 digits. We claim we can do no better than 98652. A better number must have first two digits 9 and 8. The number cannot contain 7 because $7+9=16$; hence the third digit must be 6. The next digit must be 5. Since $5+4=9$ and $6+3=9$, the last digit last digit cannot be larger than 2.

6. Seven two-digit integers form a strictly increasing arithmetic sequence. If the first and last terms of this sequence have the same set of digits, what is the sum of all possible medians of the sequence?

   Proposed by: Kevin Ren.

   Answer: $\boxed{385}$.

   The first and last integers have distinct digits and are reverses of each other. Let the first integer be $10a+b$; then the last integer is $10b+a$. The common difference $\frac{9b-9a}{6} = \frac{3(b-a)}{2}$ is an integer, so $b-a$ is even. Hence $b+a$ is even. The median is $11(a+b)/2$. Since $(a+b)/2$ ranges from 2 to 8 inclusive, the possible medians are $22, 33, \ldots, 88$, whose sum is $7 \cdot \frac{22+88}{2} = 385$.

7. Triangle ABC has $AB = 8, AC = 12, BC = 10$. Let D be the intersection of the angle bisector of angle A with BC. Let M be the midpoint of BC. The line parallel to AC passing through M intersects AB at N. The line parallel to AB passing through D intersects AC at P. MN and DP intersect at E. Find the area of $ANEP$.

   Proposed by: Brice Huang.

   Answer: $\boxed{6\sqrt{7}}$.

   Note that $ANEP$ is a parallelogram, so its area is $[ANEP] = AN \cdot AP \sin BAC$. We will compute each of these terms.

   Since N is the midpoint of AB, $AN = 4$. By properties of parallel lines and the Angle Bisector Theorem,

$$\frac{AP}{PC} = \frac{BD}{DC} = \frac{AB}{AC} = \frac{2}{3}.$$

   Thus $AP = \frac{2}{5}AC = \frac{24}{5}$.

   To compute $\sin BAC$, we compute the area $[ABC]$ two different ways. Since the semiperimeter of ABC is $\frac{1}{2}(8+10+12) = 15$, by Heron's Formula

$$[ABC] = \sqrt{15(15-AB)(15-BC)(15-CA)} = 15\sqrt{7}.$$

   But also,

$$[ABC] = \frac{1}{2}AB \cdot AC \sin BAC = 48 \sin BAC.$$

   Thus $\sin BAC = \frac{5\sqrt{7}}{16}$.

   Putting this all together, we have

$$[ANEP] = 4 \cdot \frac{24}{5} \cdot \frac{5\sqrt{7}}{16} = 6\sqrt{7}.$$

8. The Fibonacci sequence $F_0, F_1, \ldots$ satisfies $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Compute the number of triples (a, b, c) with $0 \leq a < b < c \leq 100$ for which F_a, F_b, F_c is an arithmetic progression.

   Proposed by: Ankan Bhattacharya.

   Answer: $\boxed{101}$.

   For all $b \geq 2$, $F_{b+1} > F_b$, so $F_{b+2} = F_{b+1} + F_b > 2F_b$. Thus if $b \geq 2$, F_a, F_b, F_c can only be an arithmetic progression if $c = b + 1$. Then,

$$F_a = F_b - (F_{b+1} - F_b) = F_b - F_{b-1} = F_{b-2}.$$

   If $b - 2 \geq 3$, this implies $a = b - 2$. Therefore, when $b \geq 5$ the solutions (a, b, c) are $(b - 2, b, b + 1)$, where $b \in \{5, \ldots, 99\}$. There are 95 solutions for this case.

   If $2 \leq b \leq 4$, we still must have $c = b+1$. We get the solutions $(0, 2, 3), (1, 3, 4), (2, 3, 4), (1, 4, 5), (2, 4, 5)$. If $b < 2$, we must have $a = 0$ and $b = 1$. This yields one additional solution $(0, 1, 3)$, for a total of $95 + 6 = 101$ solutions.

9. How many decreasing sequences $a_1, a_2, \ldots, a_{2019}$ of positive integers are there such that $a_1 \leq 2019^2$ and $a_n + n$ is even for each $1 \leq n \leq 2019$?

   Proposed by: Jeffery Yu.

   Answer: $\boxed{\binom{2039190}{2019}}$.

   In order for $a_n + n$ to be even, a_n and n must have the same parity. Let us define $a_0 = 2019^2 + 1, a_{2020} = 0$. Then the 2020 adjacent differences $b_i = a_{i-1} - a_i$ $(1 \leq i \leq 2020)$ are odd numbers with sum $2019^2 + 1$. Let us count the number of such $(b_1, \ldots, b_{2020})$.

   Define $b_i = 2c_i - 1$, for a positive integer c_i. Then,

$$\sum_{i=1}^{2020} b_i = \sum_{i=1}^{2020} (2c_i - 1) = 2019^2 + 1 \Rightarrow \sum_{i=1}^{2020} c_i = \binom{2020}{2} + 1.$$

   The last quantity is counted by placing 2019 dividers among the $\binom{2020}{2}$ spaces between $\binom{2020}{2} + 1$ items. Thus the number of sequences is

$$\binom{\binom{2020}{2}}{2019} = \binom{2039190}{2019}.$$

10. Let a, b be positive real numbers with $a > b$. Compute the minimum possible value of the expression

$$\frac{a^2 b - ab^2 + 8}{ab - b^2}.$$

   Proposed by: Alexander Katz.

   Answer: $\boxed{6}$.

   By AM-GM,

$$\frac{a^2 b - ab^2 + 8}{ab - b^2} = a + \frac{8}{b(a - b)} = (a - b) + b + \frac{8}{b(a - b)} \geq 3\sqrt[3]{8} = 6.$$

   Equality occurs when $a - b = b = \frac{8}{b(a-b)}$, i.e. $b = 2, a = 4$, so this minimum is attainable.

11. Let ABC be a right triangle with hypotenuse AB. Point E is on AB with $AE = 10BE$, and point D is outside triangle ABC such that $DC = DB$ and $\angle CDA = \angle BDE$. Let $[ABC]$ and $[BCD]$ denote the areas of triangles ABC and BCD. Determine the value of $\frac{[BCD]}{[ABC]}$.

   Proposed by: Kevin Ren.

   Answer: $\boxed{4}$.

Let $r = \frac{BE}{BA} = \frac{1}{11}$, $b = AC$, and x be the distance from D to BC. Let M be the midpoint of AB. Then DM bisects $\angle ADE$, so $\frac{AD}{DE} = \frac{1/2-r}{1/2} = 1 - 2r$. If the perpendicular from D to AC is ℓ, and the perpendiculars from A, E to ℓ meet ℓ at K, L respectively, then $KAD \sim LED$, so $\frac{x+rb}{x+b} = \frac{AD}{DE} = 1-2r$. Thus $x = \frac{b(1-3r)}{2r}$, so $\frac{[BCD]}{[ABC]} = \frac{x}{b} = \frac{1-3r}{2r} = \frac{8/11}{2/11} = 4$.

12. Determine the number of 10-letter strings consisting of As, Bs, and Cs such that there is no B between any two As.

Proposed by: Kevin Ren.

Answer: $\boxed{17664}$.

We do casework on the number of As.

- If there are zero As, there are $2^{10} = 1024$ valid strings.
- If there is one A, there are 10 positions for the A and 2 settings for each non-A position, for $10 \cdot 2^9 = 5120$ total valid strings.
- If there are more than two As, there are $\binom{10}{2}$ choices for the leftmost and rightmost As, and 2 settings for all positions: A or C for the positions between the leftmost and rightmost As, and B or C for the others. This gives a count of $\binom{10}{2} \cdot 2^8 = 11520$ valid strings.

This gives a final count of $1024 + 5120 + 11520 = 17664$ valid strings.

13. The infinite sequence $a_0, a_1, \ldots$ is given by $a_1 = \frac{1}{2}$, $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine the infinite product $a_1 a_2 a_3 \cdots$.

Proposed by: Brice Huang.

Answer: $\boxed{\frac{3\sqrt{3}}{4\pi}}$.

Let the sequence θ_n be such that $\cos\theta_n = a_n$ and $\theta_n \in [0, \frac{\pi}{2}]$. Then $\theta_1 = \frac{\pi}{3}$ and, by the cosine half-angle rule, $\theta_{n+1} = \frac{1}{2}\theta_n$. The desired product is

$$P = a_1 a_2 a_3 \cdots = \prod_{i=0}^{\infty} \cos\frac{\theta_1}{2^i}.$$

Consider the Nth partial product $P_N = \prod_{i=0}^{N-1} \cos\frac{\theta_1}{2^i}$. Then

$$P_N \sin\frac{\theta_1}{2^{N-1}} = \sin\frac{\theta_1}{2^{N-1}} \prod_{i=0}^{N-1} \cos\frac{\theta_1}{2^i} = \frac{\sin(2\theta_1)}{2^N}$$

by telescoping. Thus $P_N = \frac{\sin(2\theta_1)}{2^N \sin\frac{\theta_1}{2^{N-1}}}$. Note that

$$\lim_{N\to\infty} 2^{N-1} \sin\frac{\theta_1}{2^{N-1}} = \lim_{x\to 0} \frac{\theta_1 \sin x}{x} = \theta_1.$$

So, in the limit as $N \to \infty$, the denominator approaches $2\theta_1$, and the infinite product is $P = \frac{\sin(2\theta_1)}{2\theta_1}$. Plugging in $\theta_1 = \frac{\pi}{3}$ yields

$$P = \frac{\sqrt{3}/2}{2\pi/3} = \frac{3\sqrt{3}}{4\pi}.$$

14. In a circle of radius 10, three congruent chords bound an equilateral triangle with side length 8. The endpoints of these chords form a convex hexagon. Compute the area of this hexagon.

Proposed by: Kevin Ren.

Answer: $\boxed{134\sqrt{3}}$.

Let each chord's intersections with the other two chords divide it into segments of length $x, 8, x$. By equilateral triangle geometry, these intersections are $\frac{8}{\sqrt{3}}$ from the center of the circle. By Power of a Point on one of these intersections,

$$x(8+x) = 10^2 - \left(\frac{8}{\sqrt{3}}\right)^2 = 100 - \frac{64}{3}.$$

The hexagon is an equilateral triangle with side length $8+3x$ minus three equilateral triangles with side length x. Thus its area is

$$\frac{\sqrt{3}}{4}\left[(8+3x)^2 - 3x^2\right] = \frac{\sqrt{3}}{4}\left[6x^2 + 48x + 64\right] = \frac{\sqrt{3}}{4}\left[6\left(100 - \frac{64}{3}\right) + 64\right] = 134\sqrt{3}.$$

15. Let $P(x)$ be a polynomial with integer coefficients such that

$$P(\sqrt{2}\sin x) = -P(\sqrt{2}\cos x)$$

for all real numbers x. What is the largest prime that must divide $P(2019)$?

Proposed by: Brice Huang.

Answer: $\boxed{1009}$.

We first exhibit a P where 1009 is the largest prime dividing $P(2019)$. Let $P(x) = x^2 - 1$. This satisfies the condition, because

$$P(\sqrt{2}\sin x) = 2\sin^2 x - 1 = 1 - 2\cos^2 x = -P(\sqrt{2}\cos x).$$

We have $P(2019) = 2019^2 - 1 = 2018 \cdot 2020$. The largest factor of this is 1009, as desired.

Next, we show that 1009 always divides $P(2019)$. Plugging in $x = \frac{\pi}{4}$ yields $P(1) = -P(1)$, so $P(1) = 0$. Plugging in $x = \frac{3\pi}{4}$ yields $P(1) = -P(-1)$, so $P(-1) = 0$. Thus $x^2 - 1 \mid P(x)$, and $2019^2 - 1 \mid P(2019)$. Since $1009 \mid 2019^2 - 1$, we are done.

16. What is the product of the factors of 30^{12} that are congruent to 1 modulo 7?

Proposed by: Brice Huang.

Answer: $\boxed{30^{2214}}$.

First note that $30^{12} \equiv 1 \pmod 7$, so if $d \equiv 1 \pmod 7$ and d is a divisor of 30^{12}, then $\frac{30^{12}}{d} \equiv 1 \pmod 7$. Thus the geometric mean of all such factors d is 30^6. So, if N is the number of these factors, then the answer is 30^{6N}. It remains to compute N.

Each factor of 30 is of the form $2^a 3^b 5^c$, where $0 \le a, b, c \le 12$. Note that 3 is a primitive root modulo 7, and $2 \equiv 3^2 \pmod 7$, $5 \equiv 3^{-1} \equiv 3^5 \pmod 7$. Thus,

$$2^a 3^b 5^c \equiv 3^{2a+b+5c} \pmod 7,$$

and this is $1 \pmod 7$ if and only if $6 \mid 2a + b + 5c$.

For given $0 \le a, b \le 12$, there are 2 choices for c that satisfy $c \equiv 2a + b \pmod 6$, except if $6 \mid 2a + b$, in which case there are three choices. In this last case, if $a = 0, 3, 6, 9, 12$ then there are 3 choices for b; otherwise there are two. Thus in $13 \cdot 2 + 5 = 31$ cases there are three choices for c. Thus there are $13^2 \cdot 2 + 31 = 369$ valid triples (a, b, c), so $N = 369$. Thus the answer is $(30^6)^{369} = 30^{2214}$.

17. Tommy takes a 25-question true-false test. He answers each question correctly with independent probability $\frac{1}{2}$. Tommy earns bonus points for correct streaks: the first question in a streak is worth 1 point, the second question is worth 2 points, and so on. For instance, the sequence TFFTTTFT is worth $1 + 1 + 2 + 3 + 1 = 8$ points. Compute the expected value of Tommy's score.

Proposed by: Kevin Ren.

Answer: $\boxed{24 + \frac{1}{2^{25}}}$.

Let us compute the expected score Tommy earns on question n. Tommy solves question n with probability $\frac{1}{2}$, both questions $n-1$ and n with probability $\frac{1}{2^2}$, questions $n-2$ through n with probability $\frac{1}{2^3}$, and so on. By linearity of expectation, Tommy's expected score on question n is

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

By linearity of expectation again, Tommy's expected score on the test is

$$\sum_{n=1}^{25} \left(1 - \frac{1}{2^n}\right) = 25 - \left(1 - \frac{1}{2^{25}}\right) = 24 + \frac{1}{2^{25}}.$$

18. Two circles with radii 3 and 4 are externally tangent at P. Let $A \neq P$ be on the first circle and $B \neq P$ be on the second circle, and let the tangents at A and B to the respective circles intersect at Q. Given that $QA = QB$ and AB bisects PQ, compute the area of QAB.

Proposed by: Kevin Ren.

Answer: $\boxed{\frac{1008}{65}}$.

Since $QA = QB$, Q lies on the radical axis of the circles, so PQ is the common external tangent. Let M be the midpoint of PA and N be the midpoint of PB. Then MN intersects PQ at K such that $\frac{PK}{PQ} = \frac{1}{4}$. Furthermore, let O_1 be the center of the first circle and O_2 be the center of the second circle; then M lies on O_1Q and N lies on O_2Q. We also know $QMPN$ is cyclic from the right angles at M and N; thus

$$\frac{PK}{KQ} = \frac{PK}{MK} \cdot \frac{MK}{QK} = \frac{QN \cdot QM}{PN \cdot PM} = \frac{O_1P \cdot O_2P}{QP^2} = \frac{12}{QP^2}$$

by similar triangles. Thus $QP = 6$ and so the area of PAB, which is half the area of $PAQB$ and thus equal to the area of $PMQN$, is computed as $\frac{1}{2} \cdot (\frac{6}{\sqrt{5}} \cdot \frac{12}{\sqrt{5}} + \frac{12}{\sqrt{13}} \cdot \frac{18}{\sqrt{13}}) = \frac{1008}{65}$.

19. Let n be the largest integer such that 5^n divides $12^{2015} + 13^{2015}$. Compute the remainder when $\frac{12^{2015}+13^{2015}}{5^n}$ is divided by 1000.

Proposed by: Alexander Katz and Kevin Ren.

Answer: $\boxed{17}$.

By the Binomial Theorem,

$$12^{2015} + 13^{2015} = 12^{2015} + (25 - 12)^{2015} = 2015 \cdot 12^{2014} \cdot 25 - \binom{2015}{2} \cdot 12^{2013} \cdot 25^2 + \cdots,$$

where the terms afterwards are all divisible by 5^6. We see from this expansion that $n = 3$. Let us now calculate $\frac{12^{2015}+13^{2015}}{5^3}$ modulo 125 and 8.

From the above expansion, we see that

$$\frac{12^{2015} + 13^{2015}}{5^3} \equiv 403 \cdot 12^{2014} - 403 \cdot 1007 \cdot 12^{2013} \cdot 25 \pmod{125},$$

since the remaining terms in the expansion are divisible by 125. The first term can be computed as

$$403 \cdot 12^{14} \equiv 28 \cdot 1728 \cdot 144 \equiv 67 \pmod{125},$$

where we use that $12^{2000} \equiv 1 \pmod{125}$. By computing that

$$403 \cdot 1007 \cdot 12^{2013} \equiv 2 \pmod 5,$$

we can see that the second term is 50 (mod 125). Thus

$$\frac{12^{2015}+13^{2015}}{5^3} \equiv 67-50 \equiv 17 \pmod{125}.$$

In modulo 8, we can compute

$$\frac{12^{2015}+13^{2015}}{5^3} \equiv \frac{5^{2015}}{5^3} = 5^{2012} \equiv 1 \pmod{8}.$$

By the Chinese Remainder Theorem, the answer is 17.

20. Kelvin the Frog lives in the 2-D plane. Each day, he picks a uniformly random direction (i.e. a uniformly random bearing $\theta \in [0, 2\pi]$) and jumps a mile in that direction. Let D be the number of miles Kelvin is away from his starting point after ten days. Determine the expected value of D^4.

Proposed by: Brice Huang.

Answer: $\boxed{190}$.

Let $v_1, \dots, v_{10}$ denote vectors representing Kelvin's jump on each of the days. Then.

$$D^4 = ||v_1 + \ldots + v_{10}||^4 = [(v_1+\ldots+v_{10})\cdot(v_1+\ldots+v_{10})]^2 = \left[10 + 2\sum_{i<j} v_i \cdot v_j\right]^2.$$

This expands as

$$D^4 = 100 + 40\sum_{i<j} v_i \cdot v_j + 4\sum_{i<j,i'<j'} (v_i \cdot v_j)(v_{i'} \cdot v_{j'}).$$

Let $\theta_{i,j}$ denote the counterclockwise angle from vector v_i to vector v_j, so $v_i \cdot v_j = \cos\theta_{i,j}$. Thus

$$D^4 = 100 + 40\sum_{i<j} \cos\theta_{i,j} + 4\sum_{i<j,i'<j'} \cos\theta_{i,j}\cos\theta_{i',j'}.$$

The expected value of each term $\cos\theta_{i,j}$ is 0. Moreover, the expected value of each term $\cos\theta_{i,j}\cos\theta_{i',j'}$ is 0 unless $(i,j) = (i',j')$. Thus,

$$\mathbb{E}[D^4] = 100 + 4\sum_{i<j}\mathbb{E}\left[\cos^2\theta_{i,j}\right].$$

Finally, note that $\theta_{i,j}$ is uniformly distributed in $[0, 2\pi]$, so $\mathbb{E}\left[\cos^2\theta_{i,j}\right] = \frac{1}{2}$. Therefore,

$$\mathbb{E}[D^4] = 100 + 4\cdot\binom{10}{2}\cdot\frac{1}{2} = 190.$$

21. Let $ABCD$ be a rectangle satisfying $AB = CD = 24$, and let E and G be points on the extension of BA past A and the extension of CD past D respectively such that $AE = 1$ and $DG = 3$.

Suppose that there exists a unique pair of points (F, H) on lines BC and DA respectively such that H is the orthocenter of $\triangle EFG$. Find the sum of all possible values of BC.

Proposed by: Ankan Bhattacharya.

Answer: $\boxed{10\sqrt{3}}$.

Let $A(0,0)$, $B(24,0)$, $C(24,y)$, $D(0,y)$, $E(-1,0)$, $G(-3,y)$, $H(0,x)$, and assume without loss of generality that $y > 0$. Then $EG \perp FH$ implies FH has slope $\frac{2}{y}$, so $F(24, \frac{48}{y}+x)$. Also $EH \perp FG$ implies $x \cdot \frac{48/y-y+x}{27} = -1$. Let $z = \frac{48}{y} - y$; then $x(x+z) = -27$, so $x^2+xz+27 = 0$. This has exactly one solution when $z^2 = 108$, so $z = \pm 6\sqrt{3}$. Solve for $y = \pm 2\sqrt{3}, \pm 8\sqrt{3}$. The valid choices for $y = BC$ are $2\sqrt{3}$ and $8\sqrt{3}$, so the desired sum is $10\sqrt{3}$.

22. Find the largest real number λ such that

$$a_1^2+\cdots+a_{2019}^2 \geq a_1a_2+a_2a_3+\cdots+a_{1008}a_{1009}+\lambda a_{1009}a_{1010}+\lambda a_{1010}a_{1011}+a_{1011}a_{1012}+\cdots+a_{2018}a_{2019}$$

for all real numbers $a_1, \ldots, a_{2019}$. The coefficients on the right-hand side are 1 for all terms except $a_{1009}a_{1010}$ and $a_{1010}a_{1011}$, which have coefficient λ.

Proposed by: Ankan Bhattacharya and Kevin Ren.

Answer: $\boxed{\sqrt{\frac{1010}{1009}}}$.

Observe the identity

$$\begin{aligned}\sum_{i=1}^{1009} a_i^2 - \sum_{i=1}^{1008} a_ia_{i+1} &= \left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1009}^2 + \sum_{i=1}^{1008}\left(\sqrt{\frac{i+1}{2i}}a_i - \sqrt{\frac{i}{2(i+1)}}a_{i+1}\right)^2 \\ &\geq \left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1009}^2.\end{aligned}$$

This is tight when $a_i = \frac{i}{1009}a_{1009}$ for $i = 1, \ldots, 1009$, so the constant factor on a_{1009} in this bound is tight. Analogously, we have

$$\sum_{i=1011}^{2019} a_i^2 - \sum_{i=1011}^{2018} a_ia_{i+1} \geq \left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1011}^2,$$

with equality when $a_i = \frac{2020-i}{1009}a_{1011}$ for $i = 1011, \ldots, 2020$. Note that

$$\begin{aligned}\sum_{i=1}^{2019} a_i^2 - \sum_{i=1}^{1008} a_ia_{i+1} - \sum_{i=1011}^{2018} a_ia_{i+1} &\geq \left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1009}^2 + a_{1010}^2 + \left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1011}^2 \\ &= \left[\left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1009}^2 + \frac{1}{2}a_{1010}^2\right] \\ &\quad + \left[\frac{1}{2}a_{1010}^2 + \left(\frac{1}{2}+\frac{1}{2\cdot 1009}\right)a_{1011}^2\right] \\ &\geq \sqrt{\frac{1010}{1009}}a_{1009}a_{1010} + \sqrt{\frac{1010}{1009}}a_{1010}a_{1011},\end{aligned}$$

where the second bound is by AM-GM, with equality when $a_{1009} = a_{1011} = \sqrt{\frac{1009}{1010}}a_{1010}$. Thus $\lambda = \sqrt{\frac{1010}{1009}}$ satisfies the problem. Since equality is attainable for this λ, it is also optimal.

23. For Kelvin the Frog's birthday, Alex the Kat gives him one brick weighing x pounds, two bricks weighing y pounds, and three bricks weighing z pounds, where x, y, z are positive integers of Kelvin the Frog's choice.

Kelvin the Frog has a balance scale. By placing some combination of bricks on the scale (possibly on both sides), he wants to be able to balance any item of weight $1, 2, \ldots, N$ pounds. What is the largest N for which Kelvin the Frog can succeed?

Proposed by: Brice Huang.

Answer: $\boxed{52}$.

Let us first show 52 is an upper bound. Let n_x denote the number of bricks of weight x on the opposite side of the item being balanced, minus the number of bricks of weight x on the same side of the item being balanced. Similarly define n_y, n_z. Then, $n_x \in \{-1, 0, 1\}$, $n_y \in \{-2, \ldots, 2\}$, and $n_z \in \{-3, \ldots, 3\}$.

Every weight that can be balanced can be written in the form

$$n_x x + n_y y + n_z z.$$

There are $3 \cdot 5 \cdot 7 = 105$ such sums, of which at least one is 0. Moreover, if $S > 0$ can be expressed in the above form, so can $-S$. Thus there are at most $\frac{105-1}{2} = 52$ distinct positive sums. So, $N \leq 52$.

$x = 1, y = 3, z = 15$ allows Kelvin the Frog to balance any item of weight up to 52 pounds, so this bound can be attained.

24. Let ABC be a triangle with $\angle A = 60°$, $AB = 12$, $AC = 14$. Point D is on BC such that $\angle BAD = \angle CAD$. Extend AD to meet the circumcircle at M. The circumcircle of BDM intersects AB at $K \neq B$, and line KM intersects the circumcircle of CDM at $L \neq M$. Find $\frac{KM}{LM}$.

Proposed by: Kevin Ren.

Answer: $\boxed{\frac{13}{8}}$.

Extend CL to intersect AD at P. The main result is that $ACPK$ is a rhombus: we prove $AB \parallel CL$ by angle chasing, and then we show ACP is isosceles. Thus $\frac{KM}{LM} = \frac{AM}{PM}$.

Let N be the midpoint of KC. Then $CN \perp AD$, so $AN = AC \cos \frac{A}{2}$. Thus $AP = 2AC \cos \frac{A}{2}$. Let the perpendicular to AC through M meet AC at Q; then it is well-known that $AQ = \frac{AB+AC}{2}$, so $AM = \frac{AB+AC}{2 \cos \frac{A}{2}}$. Thus

$$\frac{AP}{AM} = \frac{2AC \cos \frac{A}{2}}{\frac{AB+AC}{2 \cos \frac{A}{2}}} = \frac{4AC}{AB + AC} \cos^2(A/2) = \frac{21}{13}.$$

Thus $\frac{AM}{PM} = \frac{13}{8}$ is our answer.

25. Determine the remainder when

$$\prod_{i=1}^{2016} (i^4 + 5)$$

is divided by 2017.

Proposed by: Brice Huang.

Answer: $\boxed{2013}$.

Let X denote the given expression, and S denote the set of quartic residues modulo 2017. As i ranges from 1 to 2016, i^4 attains each quartic residue four times. Thus,

$$X \equiv \left[\prod_{a \in S} (a + 5)\right]^4 \pmod{2017}.$$

The polynomial

$$P(x) = x^{504} - 1 \pmod{2017}$$

has roots precisely at the elements of S, so

$$P(x) \equiv \prod_{a \in S} (x - a) \pmod{2017}.$$

Therefore,

$$\prod_{a \in S} (a + 5) = \prod_{a \in S} (-5 - a) = P(-5) = 5^{504} - 1 \pmod{2017},$$

where the first equality uses the fact that $|S| = \frac{2016}{4} = 504$ is even. Therefore,

$$X = (5^{504} - 1)^4 \pmod{2017}.$$

By the Quadratic Reciprocity Law,

$$\left(\frac{5}{2017}\right) = \left(\frac{2017}{5}\right)(-1)^{\frac{2017-1}{2}\cdot\frac{5-1}{2}} = \left(\frac{2}{5}\right) = -1,$$

so 5 is a quadratic nonresidue modulo 2017. Thus, $5^{1008} \equiv -1 \pmod{2017}$, and

$$(5^{504}-1)^2 = 5^{1008} - 2\cdot 5^{504} + 1 \equiv -2\cdot 5^{504} \pmod{2017},$$

and

$$X \equiv (-2\cdot 5^{504})^2 \equiv 4\cdot 5^{1008} \equiv -4 \equiv 2013 \pmod{2017}.$$

26. The permutations of $OLYMPIAD$ are arranged in lexicographical order, with $ADILMOPY$ being arrangement 1 and its reverse being arrangement 40320. Yu Semo and Yu Sejmo both choose a uniformly random arrangement. The immature Yu Sejmo exclaims, "My fourth letter is L!" while Yu Semo remains silent. Given this information, let E_1 be the expected arrangement number of Yu Semo and E_2 be the expected arrangement number of Yu Sejmo. Compute $E_2 - E_1$.

Proposed by: Kevin Ren.

Answer: $\boxed{\frac{2892}{7}}$.

First, we compute the EV of Yu Sejmo. It's equivalent to random permutations σ of $(1, 2, \ldots, 8)$ with $\sigma(4) = 4$. By counting the number of arrangements before σ, we get the arrangement number of σ is $1 + \sum_{j=1}^{7}(8-j)!\sum_{k=j+1, a_j>a_k}^{8} 1$. Thus, the expected arrangement number is $1 + \sum_{j=1}^{7}(8-j)!\sum_{k=j+1}^{8}\mathbb{P}[a_j > a_k]$. The probability $\mathbb{P}[\sigma(j) > \sigma(k)]$ is $\frac{1}{2}$ except when one of j, k is 4. We also have $\mathbb{P}[\sigma(k) > \sigma(4)] = \frac{4}{7}$ when $k < 4$ and $\mathbb{P}[\sigma(4) > \sigma(k)] = \frac{3}{7}$ when $k > 4$. Aggregating our contributions gives

$$1 + \sum_{k=1}^{7}\frac{k\cdot k!}{2} + \frac{7! + 6! + 5! - 4\cdot 4!}{14}.$$

But $\sum_{k=1}^{7}\frac{k\cdot k!}{2} = \frac{8!-1}{2}$ is well-known, and the EV of Yu Semo is simply $\frac{8!+1}{2}$. Thus $E_2 - E_1 = \frac{7!+6!+5!-4\cdot 4!}{14} = \frac{2892}{7}$.

27. For an integer n, define $f(n)$ to be the greatest integer k such that 2^k divides $\binom{n}{m}$ for some $0 \le m \le n$. Compute $f(1) + f(2) + \cdots + f(2048)$.

Proposed by: Kevin Ren.

Answer: $\boxed{16409}$.

Let $v_2(n)$ denote the greatest integer k such that $2^k \mid n$. It is known that $v_2(n!) = n - s_2(n)$, where $s_2(n)$ is the number of ones in the binary representation of n. Thus

$$v_2\left(\binom{n}{m}\right) = v_2(n!) - v_2(m!) - v_2((n-m)!) = s_2(m) + s_2(n-m) - s_2(n)$$

is the number of carries needed when adding the numbers m and $n-m$ in base 2. From this result, we see that $f(n) = 0$ if the binary representation of n contains only 1s (i.e. $n = 2^a - 1$ for some a), and otherwise $f(n)$ is the number of digits before the final 0 in the binary representation of n.

Let us first compute $\sum_{n=1}^{2047} f(n)$; we will add $f(2048)$ separately. We can treat $n = 1, \ldots, 2047$ as an 11-digit binary string. Suppose in n, the leading 1 is the ath digit from the right, and the last 0 is the bth digit from the right. Then, $f(n) = b - a$. For each choice of (a, b), there are 2^{b-a-1} ways to choose the digits between the leading 1 and final 0. Thus,

$$\sum_{n=1}^{2047} f(n) = \sum_{1\le a<b\le 11}(b-a)2^{b-a-1}.$$

We compute this as follows.

$$\sum_{1\le a<b\le 11} b2^{b-a-1} = \sum_{b=2}^{11} b(2^{b-1}-1) = 10\cdot 2^{11} - 65 = 20415,$$

and

$$\sum_{1\le a<b\le 11} a2^{b-a-1} = \sum_{a=1}^{10} a(2^{11-a}-1) = 2^{12} - 79 = 4017.$$

Thus $\sum_{n=1}^{2047} f(n) = 20415 - 4017 = 16398$, and the final answer is $16398 + 11 = 16409$.

Remark 1. The general formula for $\sum_{n=1}^{2^n} f(n)$ is $(n-3)\cdot 2^n + (2n+3)$. For $n = 0, 1, 2, 3$ the values are $1, 3, 9, 27$. For $n = 4$ it is 77. This is a conspicuous example when engineer's induction fails.

28. Alex the Kat plays the following game. First, he writes the number 27000 on a blackboard. Each minute, he erases the number on the blackboard and replaces it with a number chosen uniformly randomly from its positive divisors, including itself. Find the probability that, after 2019 minutes, the number on the blackboard is 1.

Proposed by: Brice Huang. Solution by Kevin Ren.

Answer: $\boxed{\left[1 - \frac{3}{2^{2019}} + \frac{3}{3^{2019}} - \frac{1}{4^{2019}}\right]^3}$.

Note that $27000 = 2^3\cdot 3^3\cdot 5^3$. If the current number on the blackboard is $2^a\cdot 3^b\cdot 5^c$, the next number is $2^{a'}\cdot 3^{b'}\cdot 5^{c'}$, where a', b', c' are uniformly random in $\{0,\dots,a\}$, $\{0,\dots,b\}$, and $\{0,\dots,c\}$, respectively.

Consider the game where Alex initially writes 3 on the blackboard, and every minute replaces the current number (say, k) with a uniformly random number in $\{0,\dots,k\}$. The problem is equivalent to asking: if Alex plays three copies of this new game in parallel, what is the probability that after 2019 minutes, all three boards have 0 written?

Let us find the probability that a single board will never have 0 written. In the first $n-1$ numbers written ($n = 2019$), let A, B, C be the number of threes, twos, ones respectively, and let K, L be the $(n-1)$-th, n-th numbers respectively on the board. Note that $K = K(A, B, C)$ is a function of A, B, C: specifically:

- $K = 1$ if $C \ge 1$;
- $K = 2$ if $C = 0, B \ge 1$;
- $K = 3$ if $C = 0, B = 0, A \ge 1$.

In other words, K equals 1, plus 1 if $C = 0$, plus another 1 if $B = C = 0$. The probability $P(A = a, B = b, C = c, L \ne 0) = \frac{1}{4}\cdot\frac{1}{4^a}\cdot\frac{1}{3^b}\cdot\frac{1}{2^c}\cdot K(a,b,c)$.

Define $Q(a,b,c) = \frac{1}{4}\cdot\frac{1}{4^a}\cdot\frac{1}{3^b}\cdot\frac{1}{2^c}$. Writing $K = \sum_{k=1}^{K} 1$ and changing order of summation, our desired probability

$$\sum_{a+b+c=n-1,k} P(A = a, B = b, C = c, L \ne 0)$$

can be expressed as

$$\sum_{a+b+c=n-1} Q(a,b,c) + \sum_{a+b=n-1} Q(a,b,0) + Q(n-1,0,0).$$

Note that

$$Q(n-1,0,0) = \frac{1}{4^n}$$

$$\sum_{a+b=n-1} Q(a,b,0) = \frac{1}{4}\cdot\frac{\frac{1}{3^n}-\frac{1}{4^n}}{\frac{1}{3}-\frac{1}{4}} = \frac{3}{3^n} - \frac{3}{4^n}$$

$$\begin{aligned}\sum_{a+b+c=n-1} Q(a,b,c) &= \frac{1}{4}\sum_{a=0}^{n-1}\frac{1}{4^a}\cdot\frac{1}{\frac{1}{2}-\frac{1}{3}}\cdot\left(\frac{1}{2^{n-a}}-\frac{1}{3^{n-a}}\right)\\ &= \frac{3}{2}\cdot\left(\frac{\frac{1}{2^{n+1}}-\frac{1}{4^{n+1}}}{\frac{1}{2}-\frac{1}{4}}-\frac{\frac{1}{3^{n+1}}-\frac{1}{4^{n+1}}}{\frac{1}{3}-\frac{1}{4}}\right)\\ &= \frac{3}{2^n}-\frac{6}{3^n}+\frac{3}{4^n}\end{aligned}$$

Thus the probability no zero is written is $\frac{3}{2^n}-\frac{3}{3^n}+\frac{1}{4^n}$. So, the desired probability is $\left(1-\frac{3}{2^n}+\frac{3}{3^n}-\frac{1}{4^n}\right)^3$.

29. Let n be a positive integer, and let $a_1,\dots,a_n,b_1,\dots,b_n$ be real numbers. Alex the Kat writes down the n^2 numbers of the form $\min(a_i,a_j)$, and Kelvin the Frog writes down the n^2 numbers of the form $\max(b_i,b_j)$.

Let x_n be the largest possible size of the set $\{a_1,\dots,a_n,b_1,\dots,b_n\}$, such that Alex the Kat and Kelvin the Frog write down the same collection of numbers. Determine the number of distinct integers in the sequence $x_1,x_2,\dots,x_{10{,}000}$.

Proposed by: Ankan Bhattacharya.

Answer: $\boxed{11}$.

Claim 1. *x_n equals one less than the number of representations $n^2=a^2+b^2$, with $a,b\in\mathbb{Z}_{\geq 0}$.*

Proof. Let c_k be the number of solutions to $a_i \geq k$, and let d_k be the number of solutions to $b_i \leq k$. Then the number of times Alex writes down k is $c_k^2-c_{k+1}^2$, and the number of times Kelvin writes down k is $d_k^2-d_{k-1}^2$. Thus $c_k^2+d_{k-1}^2=c_{k+1}^2+d_k^2$. The quantity $c_k^2+d_{k-1}^2$ is thus constant for all k, so it equals N^2 if we take k large enough. Thus the maximum number of distinct a_i is one less than the number of possible c_k values. But the set of $\{a_i\}$ contains the set of $\{b_i\}$ since $\max(b_i,b_i)=b_i$ must be the value of some a_j, which implies the result. ☐

Let $v_p(n)$ denote the exponent of p in the prime factorization of n. By a well-known fact,

$$x_n=\prod_{p\equiv 1 \pmod 4}(2v_p(n)+1).$$

We are interested in $p=5,13,17,29,37,41,\dots$.

First, since $5\cdot 13\cdot 17\cdot 29>10000$, we can only have up to three terms in our product. To optimize, we prefer having the lowest primes, i.e. $5,13,17$.

- Case 1: one term. Then since $5^6>10000$, we get $x_n=1,3,5,7,9,11$.
- Case 2: two terms. Then since $5\cdot 13^3>10000, 5^2\cdot 13^3>10000, 5^3\cdot 13^2>10000, 5^5\cdot 13>10000$, we get $x_n=3,9,27,5,15,25,7,21$.
- Case 3: three terms. Then since $5^3\cdot 13\cdot 17>10000$, $5\cdot 13^2\cdot 17>10000$, and $5\cdot 13\cdot 17^2>10000$, we get $x_n=27,45$.

Thus the only possible values of x_n with $1\leq n\leq 10000$ are 1, 3, 5, 7, 9, 11, 15, 21, 25, 27, and 45, for an answer of 11.

30. Let ABC be a triangle with $BC=a$, $CA=b$, and $AB=c$. The A-excircle is tangent to $\overline{BC}$ at A_1; points B_1 and C_1 are similarly defined.

Determine the number of ways to select positive integers a, b, c such that

- the numbers $-a+b+c$, $a-b+c$, and $a+b-c$ are even integers at most 100, and
- the circle through the midpoints of $\overline{AA_1}$, $\overline{BB_1}$, and $\overline{CC_1}$ is tangent to the incircle of $\triangle ABC$.

Proposed by: Ankan Bhattacharya.

Answer: $\boxed{3807}$.

Let $x = s-a$, $y = s-b$, $z = s-c$, where $s = \frac{a+b+c}{2}$. Let AA_1, BB_1, CC_1 intersect at P, which has barycentric coordinates $(\frac{x}{s}, \frac{y}{s}, \frac{z}{s})$, and let the incircle ω be tangent to BC, CA, AB at D, E, F respectively. If D' is the antipode of D in ω, then it is well-known that D' is on AA_1 and $AD' = PA_1$, hence the midpoint of AA_1 is the midpoint of PD'. Similar conclusions hold for the midpoint of BB_1 and CC_1, so the circle through the midpoints of $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$ is a dilation of ω with scale factor $\frac{1}{2}$ with respect to P.

If the two circles are internally tangent, they must be internally tangent at P. Suppose P lies on minor arc $\widehat{EF}$ in ω. Since P is also on AA_1, we have $P = D'$, and so P is the midpoint of AA_1. By mass points (A has mass x and A_1 has mass $y+z$), we get $x = y+z$. Similar results hold if P lies on another minor arc in ω, in which case we get $y = x+z$ or $z = x+y$.

If the two circles are externally tangent, the condition becomes $IP = 3r$, where r is the inradius. By the barycentric distance formula,

$$-\frac{1}{4s^2}\left((s-3x)(s-3y)(x+y)^2 + (s-3y)(s-3z)(y+z)^2 + (s-3z)(s-3z)(x+z)^2\right) = 9r^2.$$

Using Heron's formula $r^2s^2 = xyzs$, we may simplify to

$$s^2\sum(x+y)^2 - 3s\sum(x+y)^3 + 9\sum xy(x+y)^2 = -36xyzs$$

where the sums are symmetric sums (e.g. $\sum x = x+y+z$, $\sum x^2y = x^2y + x^2z + \cdots + y^2z$). The identity

$$\sum xy(x+y)^2 = (x+y+z)(\sum x^2y - 2xyz)$$

allows us to cancel $s = x+y+z$ from both sides to get

$$s\sum(x+y)^2 - 3\sum(x+y)^3 + 9(\sum x^2y - 2xyz) = -36xyz.$$

By writing $\sum(x+y)^3 = 2x^3 + 2y^3 + 2z^3 + 3\sum x^2y$, we can simplify to

$$s\sum(x+y)^2 = 6(x^3+y^3+z^3-3xyz) = 6s(x^2+y^2+z^2-xy-yz-zx).$$

Cancel another s from both sides to get

$$2\sum x^2 + 2\sum xy = 6\left(\sum x^2 - \sum xy\right)$$

$$\sum x^2 = 2\sum xy$$

Using the quadratic formula to solve for z, we find this is equivalent to $\pm\sqrt{x} \pm \sqrt{y} \pm \sqrt{z} = 0$. In summary, the two circles are tangent when $\pm x \pm y \pm z = 0$ or $\pm\sqrt{x} \pm \sqrt{y} \pm \sqrt{z} = 0$. Furthermore, our conditions on x, y, z give $1 \le x, y, z \le 50$.

Suppose ABC is scalene and assume WLOG that $x < y < z$. If $x + y = z$ then we get $1+1+2+2+\cdots+24+24 = 600$ cases. If $\sqrt{x}+\sqrt{y} = \sqrt{z}$ then $(x:y:z)$ can be one of:

$$(1:4:9), (1:9:16), (1:16:25), (1:25:36), (1:36:49), (4:9:25), (4:24:49), (9:16:49).$$

They correspond to $5, 3, 2, 1, 1, 2, 1, 1$ cases respectively. Thus there are 616 solutions (x, y, z) with $x < y < z$. Removing the assumption that $x < y < z$, we get $616 \cdot 6 = 3696$ solutions for scalene ABC.

Suppose ABC is isosceles. Assume WLOG that $x = y < z$. If $x + y = z$ then we get 25 solutions. If $\sqrt{x}+\sqrt{y} = \sqrt{z}$ then $z = 4x$, which yields 12 solutions. By symmetry, there are a total of $3(25+12) = 111$ solutions where ABC is isosceles.

The final answer is $3696 + 111 = 3807$.