USMCA National Championship 2020

Premier Division

April 26 — May 9, 2020

Time Limit: 3 hours.
Each problem is worth 7 points.

1. Let $\mathcal{P}$ be a finite set of squares on an infinite chessboard. Kelvin the Frog notes that $\mathcal{P}$ may be tiled with only 1×2 dominoes, while Alex the Kat notes that $\mathcal{P}$ may be tiled with only 2×1 dominoes. The dominoes cannot be rotated in each tiling. Prove that the area of $\mathcal{P}$ is a multiple of 4.

2. Let ABC be an acute triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Show that $BCHG$ is a parallelogram.

3. Call a polynomial f with positive integer coefficients *triangle-compatible* if any three coefficients of f satisfy the triangle inequality. For instance, $3x^3 + 4x^2 + 6x + 5$ is triangle-compatible, but $3x^3 + 3x^2 + 6x + 5$ is not. Given that f is a degree 20 triangle-compatible polynomial with -20 as a root, what is the least possible value of $f(1)$?

4. Suppose $n > 1$ is an odd integer satisfying $n \mid 2^{\frac{n-1}{2}} + 1$. Prove or disprove that n is prime.

   Note: unfortunately, the original form of this problem did not include the red text, rendering it unsolvable. We sincerely apologize for this error and are taking concrete steps to prevent similar issues from reoccurring, including computer-verifying problems where possible. All teams will receive full credit for the question.

5. Alex the Kat and Kelvin the Frog play a game on a complete graph with n vertices. Kelvin goes first, and the players take turns selecting either a single edge to remove from the graph, or a single vertex to remove from the graph. Removing a vertex also removes all edges incident to that vertex. The player who removes the final vertex wins the game. Assuming both players play perfectly, for which positive integers n does Kelvin have a winning strategy?

6. Let P be a non-constant polynomial with integer coefficients such that if n is a perfect power, so is $P(n)$. Prove that $P(x) = x$ or P is a perfect power of a polynomial with integer coefficients.

   A perfect power is an integer n^k, where $n \in \mathbb{Z}$ and $k \geq 2$. A perfect power of a polynomial is a polynomial $P(x)^k$, where P has integer coefficients and $k \geq 2$.

7. Let $ABCD$ be a convex quadrilateral, and let ω_A and ω_B be the incircles of $\triangle ACD$ and $\triangle BCD$, with centers I and J. The second common external tangent to ω_A and ω_B touches ω_A at K and ω_B at L. Prove that lines AK, BL, IJ are concurrent.

8. Let n, m be positive integers, and let α be an irrational number satisfying $1 < \alpha < n$. Define the set

$$X = \{a + b\alpha : 0 \leq a \leq n \text{ and } 0 \leq b \leq m\}.$$

   Let $x_0 \leq x_1 \leq \cdots \leq x_{(n+1)(m+1)-1}$ be the elements of X. Show that for all $i + j \leq (n+1)(m+1) - 1$, we have that $x_{i+j} \leq x_i + x_j$.