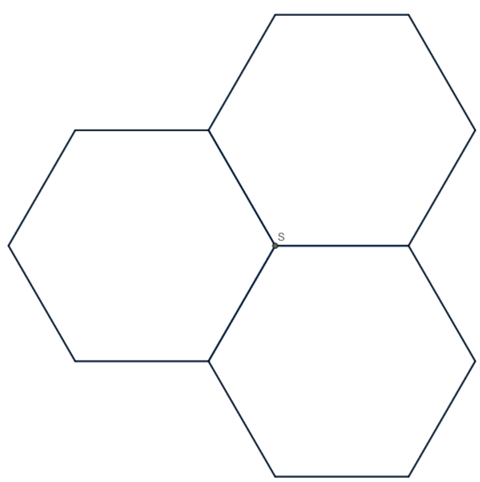
Team ______________________________ Team ID# ____________

USMCA National Finals 2021: Challenger

Saturday, April 24, 2021

1. Let $a_1, a_2, \ldots, a_{2021}$ be a sequence, where each a_i is a positive factor of 2021. How many possible values are there for the product $a_1 a_2 \cdots a_{2021}$?
2. A four-digit positive integer is called *doubly* if its first two digits form some permutation of its last two digits. For example, 1331 and 2121 are both *doubly*. How many four-digit *doubly* positive integers are there?
3. Let $f(n)$ be a sequence of integers defined by $f(1) = 1, f(2) = 1$, and $f(n) = f(n-1) + (-1)^n f(n-2)$ for all integers $n \geq 3$. What is the value of $f(20) + f(21)$?
4. I roll three special six-sided dice. Each die has faces labeled U, S, M, C, A, or *. The star can represent any of U, S, M, C, A. What is the probability that I can arrange the dice to spell out USA? (For instance, A*U is valid, but UU* is not valid.)
5. Let A denote the set of all the positive integer divisors of 30. For each nonempty subset $s \subseteq A$, define $p(s)$ to be the product of the elements in s. Finally, let B denote the set of all possible remainders when $p(s)$ is divided by 30. How many (distinct) elements are in B?
6. Let $ABCD$ be a unit square. Construct point E outside $ABCD$ such that $\overline{AE} = \sqrt{2} \cdot \overline{BE}$ and $\angle AEB = 135°$. Also, let F be the foot of the perpendicular from A to line BE. Find the area of $\triangle BDF$.
7. Find the expected value of $\max(\min(a,b), \min(c,d), \min(e,f))$ over all permutations (a,b,c,d,e,f) of $(1,2,3,4,5,6)$.
8. Let $ABCD$ be a parallelogram with $AB = CD = 16$ and $BC = AD = 24$. Suppose the angle bisectors of $\angle A$ and $\angle D$ intersect BC at E and F, respectively. Moreover, suppose AE and DF intersect at P. Given that the sum of the areas of quadrilaterals $ABFP$ and $DCEP$ is 100, compute the area of the parallelogram.
9. For how many two-digit integers n is $13 \mid 1 - 2^n - 3^n + 5^n$?
10. Find the sum of all positive integers $n \leq 1000$ with the property that for every prime number p dividing n, we have that $2p - 1$ also divides n.
11. Let $f_1(x) = x^2 - 3$ and $f_n(x) = f(f_{n-1}(x))$ for $n \geq 2$. Let m_n be the smallest positive root of f_n, and M_n be the largest positive root of f_n. If x is the least number such that $M_n \leq m_n \cdot x$ for all $n \geq 1$, compute x^2.
12. Find the sum of the three smallest positive integers N such that N has a units digit of 1, N^2 has a tens digit of 2, and N^3 has a hundreds digit of 3.
13. An ant is currently located in the center (vertex S) of the adjoined hexagonal configuration, as shown in the figure below. Each minute, it walks along 1 of the 15 possible edges, traveling from one vertex to another. How many ways are there for the ant to be back to its original position after 2020 minutes?



14. Derek the Dolphin and Kevin the Frog are playing a game where they take turns taking coins from a stack of N coins, except with one rule: The number of coins someone takes each turn must be a positive power of 6. The person who cannot take any more coins loses. If Derek goes first, how many integers N from 1 to 6^{2021} inclusive will guarantee him a win?

15. Find the sum of all real values of A such that the equation $Axy + 25x^2 + 25y^2 - 20x - 22y + 5 = 0$ has a unique solution in real numbers (x, y).

16. Let $\mathcal{C}$ be a right circular cone with height $\sqrt{15}$ and base radius 1. Let V be the vertex of $\mathcal{C}$, B be a point on the circumference of the base of $\mathcal{C}$, and A be the midpoint of VB. An ant travels at constant velocity on the surface of the cone from A to B and makes two complete revolutions around $\mathcal{C}$. Find the distance the ant travelled.

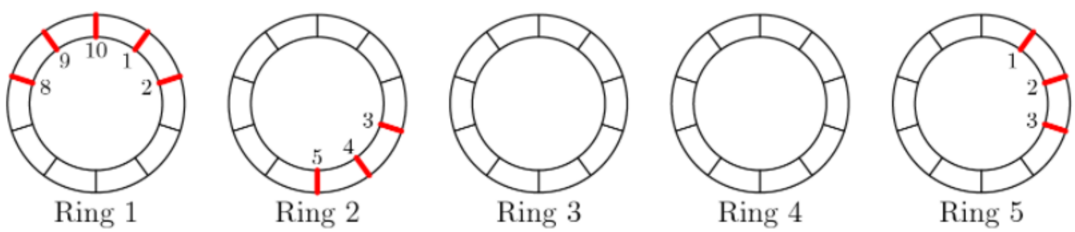
17. Let $X_1X_2X_3X_4$ be a quadrilateral inscribed in circle Ω such that $\triangle X_1X_2X_3$ has side lengths $13, 14, 15$ in some order. For $1 \le i \le 4$, let l_i denote the tangent to Ω at X_i, and let Y_i denote the intersection of l_i and l_{i+1} (indices taken modulo 4). Find the least possible area of $Y_1Y_2Y_3Y_4$.

18. Charlie has a fair n-sided die (with each face showing a positive integer between 1 and n inclusive) and a list of n consecutive positive integer(s). He first rolls the die and if the number showing on top is k, he then uniformly and randomly takes a k-element subset from his list and calculates the sum of the numbers in his subset. Given that the expected value of this sum is 2020, compute the sum of all possible values of n.

19. Let ABC be an equilateral triangle with unit side length and circumcircle Γ. Let D_1, D_2 be the points on Γ such that $BD_i = 3CD_i$. Let E_1, E_2 be the points on Γ such that $CE_i = 3AE_i$. Let F_1, F_2 be the points on Γ such that $AF_i = 3BF_i$. Then points $D_1, D_2, E_1, E_2, F_1, F_2$ are the vertices of a convex hexagon. What is the area of this hexagon?

20. Let $\tau(n)$ be the number of positive divisors of n, let $f(n) = \sum_{d|n} \tau(d)$, and let $g(n) = \sum_{d|n} f(d)$. Let P_n be the product of the first n prime numbers, and let $M = P_1P_2 \cdots P_{2021}$. Then $\sum_{d|M} \frac{1}{g(d)} = \frac{a}{b}$, where a, b are relatively prime positive integers. What is the remainder when $\tau(ab)$ is divided by 2017? (Here, $\sum_{d|n}$ means a sum over the positive divisors of n.)

21. Sarah has five rings (numbered 1 through 5), each with ten rungs labeled 1 through 10. Rung i is adjacent to rung $i+1$ for $1 \le i \le 9$, and rung 10 is adjacent to rung 1. How many ways can Sarah paint some (possibly none) of the rungs red such that in each ring, the red rungs form a contiguous block, and the total number of red rungs across the five rings is divisible by 11? (For example, Sarah can paint rungs $8, 9, 10, 1, 2$ on ring 1, rungs $3, 4, 5$ on ring 2, no rungs on rings 3 and 4, and rungs $1, 2, 3$ on ring 5.)



22. Let ABC be a triangle with $AB = 20$, $AC = 21$, and $\angle BAC = 90^\circ$. Suppose Γ_1 is the unique circle centered at B and passing through A, and Γ_2 is the unique circle centered at C and passing through A. Points E and F are selected on Γ_1 and Γ_2, respectively, such that E, A, F are collinear in that order. The tangent to Γ_1 at E and the tangent to Γ_2 at F intersect at P. Given that $PA \perp BC$, compute the area of PBC.

23. Given real numbers x, y, z, w such that $(x+y+2z)(x+z+3w) = 1$, what is the minimum possible value of $x^2+y^2+z^2+w^2$?

24. The center cell of a 5×5 square grid is removed. Determine the number of ways to color the remaining 24 cells one of four colors (cyan, magenta, yellow, and black) such that any 2×2 square of cells not containing the center cell contains cells of all four colors.

25. Convex equiangular hexagon $ABCDEF$ has $AB = CD = EF = \sqrt{3}$ and $BC = DE = FA = 2$. Points X, Y, and Z are situated outside the hexagon such that AEX, ECY, and CAZ are all equilateral triangles. Compute the area of the region bounded by lines XF, YD, and ZB.

26. How many pairs of integers (a, b) satisfy $1 \le a < 1001^3$, $1 \le b < 1001^2$, and $1001^3 \mid a^3 + ab$?

27. You are participating in a virtual stock market, with many different stocks. For a stock S, there is a list of prices where the ith number is the price of the stock on day i. On each day i, you are given the stock's current price (in dollars), and you can either buy a share of stock S, sell your share of stock S, or do nothing, but you may only take one of these actions per day, and you may not have more than one share of stock S at a time. Each stock is independent, so for example on the first day, you may buy a share of S and a share of T, and on the second day you may sell your share of T.

    At USMCA Trading LLC, you are given 2021! different stocks, where each stock's list of prices corresponds to a unique permutation of the first 2021 positive integers, to trade for 2021 days. You start out with M dollars, and at the end of 2021 days, you end up with N dollars. Assume M is large enough so that you can never run out of money during the 2021 days. What is the maximum possible value of $N - M$?

28. How many functions $f : \mathbb{Z} \to \{0, 1, 2, \cdots, 2020\}$ are there such that $f(n) = f(n+2021)$ and $2021 \mid f(2n) - f(n) - f(n-1)$ for all integers n?

29. Three circles $\Gamma_A, \Gamma_B, \Gamma_C$ are externally tangent. The circles are centered at A, B, C and have radii $4, 5, 6$ respectively. Circles Γ_B and Γ_C meet at D, circles Γ_C and Γ_A meet at E, and circles Γ_A and Γ_B meet at F. Let GH be a common external tangent of Γ_B and Γ_C on the opposite side of BC as EF, with G on Γ_B and H on Γ_C. Lines FG and EH meet at K. Point L is on Γ_A such that $\angle DLK = 90^\circ$. Compute $\frac{LG}{LH}$.

30. I start with a sequence of letters $A_1A_2\cdots A_{2021}A_1A_2\cdots A_{2021}A_1A_2\cdots A_{2021}$. I go through $i = 1, 2, 3, \cdots, 6062$ in order, and for each i, I can choose to swap letters i and $i+1$. Let N be the number of distinct strings I can end up with. What is the remainder when N is divided by 2017?