Team ______________________________ Team ID# ____________

USMCA National Finals 2021: Challenger

Saturday, April 24, 2021

1. Let $a_1, a_2, \ldots, a_{2021}$ be a sequence, where each a_i is a positive factor of 2021. How many possible values are there for the product $a_1 a_2 \cdots a_{2021}$?

   Proposed by: Kevin Ren

   Answer: $\boxed{2022^2}$

   Each a_i is $1, 43, 47$, or $43 \cdot 47$. Thus, the product $a_1 a_2 \cdots a_{2021} = 43^m 47^n$ where $0 \le m, n \le 2021$. As a result, there are 2022^2 possible values for the product.

2. A four-digit positive integer is called *doubly* if its first two digits form some permutation of its last two digits. For example, 1331 and 2121 are both *doubly*. How many four-digit *doubly* positive integers are there?

   Proposed by: Kyle Lee

   Answer: $\boxed{171}$

   Note that such an integer can have at most two distinct digits, so consider the set of two digits. If they are both nonzero and distinct, then there are $\binom{9}{2} \cdot 4 = 144$ such numbers. If they are both nonzero but the same, then there are 9 such numbers. Lastly, if one of the digits is 0, then there are $9 \cdot 2 = 18$ such numbers. The requested answer is $144 + 9 + 18 = 171$.

3. Let $f(n)$ be a sequence of integers defined by $f(1) = 1, f(2) = 1$, and $f(n) = f(n-1) + (-1)^n f(n-2)$ for all integers $n \ge 3$. What is the value of $f(20) + f(21)$?

   Proposed by: Kyle Lee

   Answer: $\boxed{89}$

   We have

   $$f(2n) = f(2n-1) + f(2n-2) = 2f(2n-2) - f(2n-3) = f(2n-2) + f(2n-4),$$

   and $f(2) = 1$, $f(3) = 0$, $f(4) = 1$, so $f(2n)$ is just the nth Fibonacci number. Therefore, the answer is $f(20) + f(21) = f(22) = F_{11} = 89$.

4. I roll three special six-sided dice. Each die has faces labeled U, S, M, C, A, or *. The star can represent any of U, S, M, C, A. What is the probability that I can arrange the dice to spell out USA? (For instance, A*U is valid, but UU* is not valid.)

   Proposed by: Kevin Ren

   Answer: $\boxed{\frac{17}{108}}$

   We casework on the number of stars on the dice.

   Three stars: There is one valid way to have three stars.

   Two stars: There are three choices for the die containing the non-star, and three choices for the non-star (U, S, or A), for a total of $3 \cdot 3 = 9$ choices.

   One star: There are three choices for the die containing the star, and $3 \cdot 2 = 6$ ways to choose two distinct letters for the non-star dice, for a total of $3 \cdot 6 = 18$ choices.

   Zero stars: There are $3! = 6$ ways to arrange U, S, A on the three dice in some order.

   There are $1 + 9 + 18 + 6 = 34$ cases. Dividing by the total number of possibilities, $6^3 = 216$, we arrive at an answer of $\frac{17}{108}$.

5. Let A denote the set of all the positive integer divisors of 30. For each nonempty subset $s \subseteq A$, define $p(s)$ to be the product of the elements in s. Finally, let B denote the set of all possible remainders when $p(s)$ is divided by 30. How many (distinct) elements are in B?

   Proposed by: Kyle Lee

   Answer: $\boxed{11}$

   The factors of 30 are $1, 2, 3, 5, 6, 10, 15, 30$. Let

   $$A = \{1, 30\}, B = \{2, 15\}, C = \{3, 10\}, D = \{5, 6\}.$$

   Note that we can pick at most one number from each set to avoid overcounting 0 mod 30.

   Numbers from only one set: $\{0, 1, 2, 3, 5, 6, 10, 15\}$. There are $\tau(30) = 8$ of them. Ignore A from this point forward.

   Numbers from two sets and not in case above: $\{12, 18, 20\}$. There are 3 here.

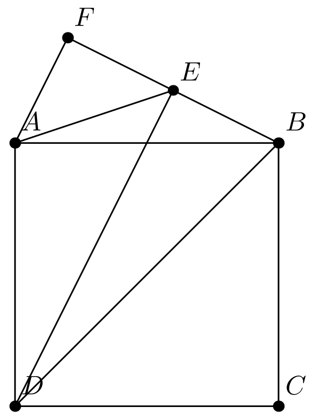
   Numbers from three sets and not in cases above: $\emptyset$. There are only 3 cases to check as any product with factors of $2, 3, 5$ will be 0 mod 30.

   Hence, the answer is $8 + 3 = 11$.

6. Let $ABCD$ be a unit square. Construct point E outside $ABCD$ such that $\overline{AE} = \sqrt{2} \cdot \overline{BE}$ and $\angle AEB = 135^\circ$. Also, let F be the foot of the perpendicular from A to line BE. Find the area of $\triangle BDF$.

Proposed by: Derek Zhu

Answer: $\boxed{\frac{3}{5}}$



Solution 1. We want to calculate the expression $\frac{1}{2} \cdot \overline{BD} \cdot \overline{BF} \cdot \sin \angle DBF$. $\overline{BD} = \sqrt{2}$ as it's the diagonal of the square, and $\overline{BF} = 1 \cdot \cos(\angle ABF)$ in $\triangle ABF$. Also, $\angle DBF = \angle ABF + 45^\circ$. Set $BE = x$. Then $AE = x\sqrt{2}$ and AEF is a $45-45-90$ right triangle, so $AF = FE = x$. By Pythagorean theorem on ABF, we see that $x^2 + (2x)^2 = 1$, so $x = \frac{1}{\sqrt{5}}$. Furthermore, $\sin \angle ABF = x = \frac{1}{\sqrt{5}}$ and $BF = 2x = \frac{2}{\sqrt{5}}$. Thus, $\sin \angle DBF = \sin \angle ABF + 45^\circ = (\sin \angle ABF + \cos \angle ABF) \cdot \frac{\sqrt{2}}{2} = \frac{3}{\sqrt{10}}$. So our final expression is $\frac{1}{2} \cdot \sqrt{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} = \frac{3}{5}$.

Solution 2. As in solution 1, we compute $BE = \frac{1}{\sqrt{5}}$ and $BF = \frac{2}{\sqrt{5}}$. Let Γ be the circumcircle of $ABCD$. By inscribed angles, we see that E lies on Γ, and $\angle BED = 90^\circ$. Thus, by Pythagorean theorem, we see that $BD = \sqrt{2}$ and $DE = \frac{3}{\sqrt{5}}$. The area of BDF is $\frac{1}{2} BF \cdot DE = \frac{3}{5}$.

7. Find the expected value of $\max(\min(a,b), \min(c,d), \min(e,f))$ over all permutations (a,b,c,d,e,f) of $(1,2,3,4,5,6)$.

Proposed by: Derek Zhu

Answer: $\boxed{\frac{19}{5}}$

If we tried to find $\max(\min(w,x), \min(y,z))$, where (w,x,y,z) is a permutation of (c,d,e,f), then there's a $\frac{2}{3}$ chance it's d and a $\frac{1}{3}$ chance it's e, as this solely depends on whether c and d are paired together or not.

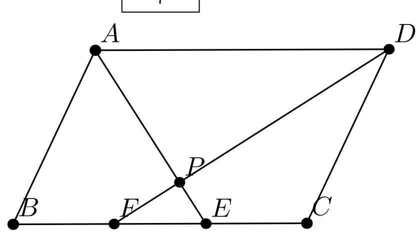
WLOG $a = 1$ and do casework on b: If $b = 2$, answer is $4 \cdot \frac{2}{3} + 5 \cdot \frac{1}{3} = \frac{13}{3}$. If $b = 3$, answer is $4 \cdot \frac{2}{3} + 5 \cdot \frac{1}{3} = \frac{13}{3}$. If $b = 4$, answer is $3 \cdot \frac{2}{3} + 5 \cdot \frac{1}{3} = \frac{11}{3}$. If $b = 5$, answer is $3 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{10}{3}$. If $b = 6$, answer is $3 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{10}{3}$.

Then taking the average, our final answer is $(\frac{13}{3} + \frac{13}{3} + \frac{11}{3} + \frac{10}{3} + \frac{10}{3})/5 = \frac{19}{5}$.

8. Let $ABCD$ be a parallelogram with $AB = CD = 16$ and $BC = AD = 24$. Suppose the angle bisectors of $\angle A$ and $\angle D$ intersect BC at E and F, respectively. Moreover, suppose AE and DF intersect at P. Given that the sum of the areas of quadrilaterals $ABFP$ and $DCEP$ is 100, compute the area of the parallelogram.

Proposed by: Kyle Lee

Answer: $\boxed{\frac{1200}{7}}$



Note that both $\angle BAE = \angle BEA$ and $\angle CDF = \angle CFD$ together implies that $BF = FE = EC = 8$, so it follows that $ABFP$ and $DCEP$ have equal areas. If we let h be the distance from P to BC, then using $EPF \sim APD$, we see that $4h$ is the vertical distance between AD and BC. Thus, we have

$$\frac{8h}{16(4h)} = \frac{[FPE]}{50 + [FPE]},$$

so $[FPE] = \frac{50}{7}$. Then the area of the parallelogram is simply $(3^2 + 1)(\frac{50}{7}) + 100 = \frac{1200}{7}$.

9. For how many two-digit integers n is $13 \mid 1 - 2^n - 3^n + 5^n$?

Proposed by: Kevin Ren

Answer: $\boxed{46}$

Since $2^n \equiv 15^n \pmod{13}$, we get $13 \mid (3^n - 1)(5^n + 1)$, so n is 2 mod 4 or divisible by 3. There are 23 two-digit integers that are 2 mod 4, 30 two-digit integers that are divisible by 3, and 7 two-digit integers that are 2 mod 4 and divisible by 3. By the Principle of Inclusion-Exclusion, there are $23 + 30 - 7 = 46$ such integers.

10. Find the sum of all positive integers $n \leq 1000$ with the property that for every prime number p dividing n, we have that $2p - 1$ also divides n.

Proposed by: Kyle Lee

Answer: $\boxed{5806}$

Clearly $n = 1$ is a solution. Assume $n > 1$.

If p is a prime divisor of n, then $p(2p - 1)$ divides n. Thus, $p(2p - 1) \leq 1000$, which forces $p \leq 19$.

If $7 \mid n$, then $13 \mid n$, $25 \mid n$, which means $n \geq 7 \cdot 13 \cdot 25 > 1000$. 1

If $11 \mid n$, then $21 \mid n$, $7 \mid n$, $13 \mid n$, which means $n \geq 11 \cdot 13 \cdot 21 > 1000$.

If $13 \mid n$, then $25 \mid n$, $9 \mid n$, which means $n \geq 13 \cdot 25 \cdot 9 > 1000$.

If $17 \mid n$, then $33 \mid n$, $11 \mid n$, which is bad from above.

If $19 \mid n$, then $37 \mid n$, $73 \mid n$, which means $n \geq 19 \cdot 37 \cdot 73 > 1000$.

Thus, the largest prime factor of n is 5. From here, it is easy to see that either either $n = 2 \cdot 3^2 \cdot 5 \cdot a$, where a contains no prime factors other than $2, 3, 5$ or $n = 3^2 \cdot 5 \cdot b$, where b contains no prime factors other than $3, 5$. Then, the answer is $90 + 180 + 270 + 360 + 450 + 540 + 720 + 810 + 900 + 45 + 135 + 225 + 405 + 675 + 1 = 5806$.

11. Let $f_1(x) = x^2 - 3$ and $f_n(x) = f_1(f_{n-1}(x))$ for $n \geq 2$. Let m_n be the smallest positive root of f_n, and M_n be the largest positive root of f_n. If x is the least number such that $M_n \leq m_n \cdot x$ for all $n \geq 1$, compute x^2.

Proposed by: Kevin Ren

Answer: $\boxed{4 + \sqrt{13}}$

The positive roots of f_n are $\sqrt{3 \pm \sqrt{3 \pm \sqrt{3 \pm \cdots \pm \sqrt{3}}}}$, where there are n square roots. To get M_n, we take all plus signs. To get m_n, we take the first sign to be minus, and all remaining signs to be plus.

We also compute $x = \sqrt{3 + \sqrt{3 + \cdots}}$ as follows: we have $x = \sqrt{3 + x}$, so $x^2 - x - 3 = 0$, and $x = \frac{1+\sqrt{13}}{2}$.

Thus, $x = \frac{\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}}{\sqrt{3-\sqrt{3+\sqrt{3+\cdots}}}} = \frac{\sqrt{3+\frac{1+\sqrt{13}}{2}}}{\sqrt{3-\frac{1+\sqrt{13}}{2}}} = \sqrt{4 + \sqrt{13}}$.

12. Find the sum of the three smallest positive integers N such that N has a units digit of 1, N^2 has a tens digit of 2, and N^3 has a hundreds digit of 3.

Proposed by: Kyle Lee

Answer: $\boxed{1883}$

Let $N = 100a + 10b + 1$, where $a \geq 0$ and b is a digit. Then

$$N^2 \equiv (10b + 1)^2 \equiv 20b + 1 \pmod{100}.$$

Since N^2 has tens digit 2, we must have $b = 1$ or $b = 6$.

Case 1. $b = 1$. Then

$$N^3 = (100a + 11)^3 \equiv 300a + 11^3 \equiv 300a + 331 \pmod{1000}.$$

Since N^3 has hundreds digit 3, a must have units digit 0.

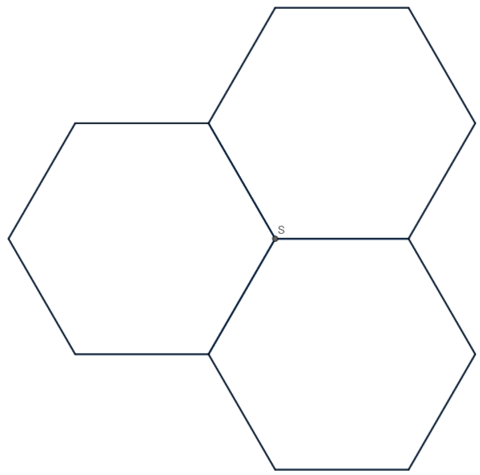
Case 2. $b = 6$. Then

$$N^3 = (100a + 61)^3 \equiv 300a + 61^3 \equiv 300a + 981 \pmod{1000}.$$

Since N^3 has hundreds digit 3, a must have units digit 8.

The three smallest values of N are then $11, 861$, and 1011, for an answer of $11 + 861 + 1011 = 1883$.

13. An ant is currently located in the center (vertex S) of the adjoined hexagonal configuration, as shown in the figure below. Each minute, it walks along 1 of the 15 possible edges, traveling from one vertex to another. How many ways are there for the ant to be back to its original position after 2020 minutes?



Proposed by: Kyle Lee

Answer: $\boxed{\frac{1}{5}(3+2\cdot 6^{1010})}$

Relabel the vertices such that $A = S$ is the center, B is the set of vertices with distance 1 from A, D is the set of vertices with distance 2 from A, and C is the set of vertices with distance 3 from A. Then we have the following recursion:

$$\begin{aligned} A_n &= B_{n-1} + B_{n-1} + B_{n-1} \\ B_n &= D_{n-1} + D_{n-1} + A_{n-1} \\ C_n &= D_{n-1} + D_{n-1} \\ D_n &= C_{n-1} + B_{n-1}. \end{aligned}$$

It suffices to find A_{2020}. Now, we have

$$\begin{aligned} A_n &= 3B_{n-1} \\ &= 3(A_{n-2} + C_{n-1}) \\ &= 3A_{n-2} + 3C_{n-1} \\ &= 3A_{n-2} + 6D_{n-2} \\ &= 3A_{n-2} + 6C_{n-3} + 6B_{n-3} \\ &= 3A_{n-2} + 6(4D_{n-4} + A_{n-4}) \\ &= 3A_{n-2} + 6A_{n-4} + 24D_{n-4} \\ &= 3A_{n-2} + 6A_{n-4} + 4A_{n-2} - 12A_{n-4} \\ &= 7A_{n-2} - 6A_{n-4}. \end{aligned}$$

Now, define the new sequence $b_n = A_{2n}$ so that we have $b_n = 7b_{n-1} - 6b_{n-2}$. The characteristic polynomial is $\lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1)$, so we have $b_n = p \cdot 1^n + q \cdot 6^n$. It is easy to see that $b_0 = 1$ and $b_1 = 3$, so we have $(p, q) = (\frac{3}{5}, \frac{2}{5})$. Then $A_{2020} = b_{1010} = \dfrac{3 + 2 \cdot 6^{1010}}{5}$.

14. Derek the Dolphin and Kevin the Frog are playing a game where they take turns taking coins from a stack of N coins, except with one rule: The number of coins someone takes each turn must be a power of 6. The person who cannot take any more coins loses. If Derek goes first, how many integers N from 1 to 6^{2021} inclusive will guarantee him a win? (Example: If $N = 37$, then a valid sequence of moves is: Derek takes one coin, Kevin takes 36 coins, and Kevin wins.)

Proposed by: Derek Zhu

Answer: $\boxed{\frac{4(6^{2021}+1)}{7} \text{ (or } \frac{4(6^{2021}-6)}{7} + 1)}$

We conclude that $N \equiv 0, 2, 4 \pmod 7$ seem to be losses and $N \equiv 1, 3, 5, 6 \pmod 7$ seem to be wins. We can prove this by induction. For $N \leq 5$, everyone is forced to only take 1 coin per turn and Derek wins if N is odd and loses if N is even. For $N = 6$, Derek takes 6 coins and wins. For general $N > 6$, if $N \equiv 1, 3, 5 \pmod 7$, Derek can take one coin and force Kevin into a losing position $(0, 2, 4 \pmod 7)$. If $N \equiv 6 \pmod 7$, Derek can take 6 coins and force Kevin into a losing position $(0 \pmod 7)$. If $N \equiv 0, 2, 4 \pmod 7$, whether Derek takes a number of coins that are 1 or 6 $\pmod 7$, Kevin can take the number of coins that's 6 or 1 (mod 7) and Derek is back in a losing position after Kevin's turn $(0, 2, 4 \pmod 7)$.

(Note: due to an ambiguity in the problem, we are also accepting an answer of $\frac{4(6^{2021}-6)}{7}+1$*, which corresponds to the problem where it's not allowed to take one coin. This new problem can be solved in the same way, except we should consider mod 42 instead of mod 7.)*

15. Find the sum of all real values of A such that the equation $Axy + 25x^2 + 25y^2 - 20x - 22y + 5 = 0$ has a unique solution in real numbers (x, y).

Proposed by: Derek Zhu

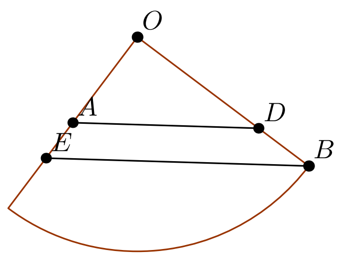
Answer: $\boxed{88}$

The values are $A = 40$ and $A = 48$. Rearrange the equation to get $25x^2 + (Ay - 20)x + (25y^2 - 22y + 5) = 0$, and let discriminant with respect to x equal to zero (as the equation must have a unique solution (x, y)) to get $(Ay - 20)^2 = 4 \cdot 25 \cdot (25y^2 - 22y + 5)$. Rearrange this to get $(A^2 - 2500)y^2 + (-40A + 2200)y - 100 = 0$, and let the discriminant with respect to y equal to zero to get $(A - 40)(A - 48) = 0$. Thus, the sum of all values of A is $40 + 48 = 88$.

16. Let $\mathcal{C}$ be a right circular cone with height $\sqrt{15}$ and base radius 1. Let V be the vertex of $\mathcal{C}$, B be a point on the circumference of the base of $\mathcal{C}$, and A be the midpoint of VB. An ant travels at constant velocity on the surface of the cone from A to B and makes two complete revolutions around $\mathcal{C}$. Find the distance the ant travelled.

Proposed by: Kevin Ren

Answer: $\boxed{2\sqrt{3} + 2\sqrt{6}}$

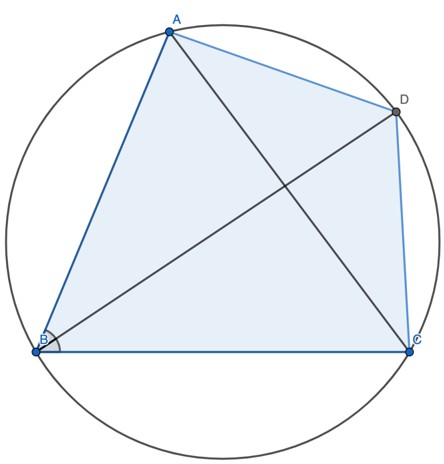


Unwrap the cone so that it becomes a circular sector with radius $\sqrt{1^2 + (\sqrt{15})^2} = 4$ and angle $\theta = \frac{\pi}{2}$ because the sector's circumference is $2\theta \cdot 4 = 2\pi \cdot 1$. The sector unwrapping consists of two segments and an arc; let A be the midpoint of one segment and B be on the other segment. Let D be the point on the same segment as B that the ant hits after the first revolution and E be that corresponding point on the same segment as A. Since $\triangle VAD$ and $\triangle VEB$ are similar right triangles, we have $\frac{VA}{VD} = \frac{VE}{VB}$ and $VA = 2$ and $VB = 4$, so $VD = VE = 2\sqrt{2}$. Then since $\angle AVD = \angle EVB = \frac{\pi}{2}$, we know $AD = 2\sqrt{3}$ and $EB = 2\sqrt{6}$. The ant travelled distance $AD + EB$, which is $2\sqrt{3} + 2\sqrt{6}$.

17. Let $X_1X_2X_3X_4$ be a quadrilateral inscribed in circle Ω such that $\triangle X_1X_2X_3$ has side lengths 13, 14, 15 in some order. For $1 \le i \le 4$, let l_i denote the tangent to Ω at X_i, and let Y_i denote the intersection of l_i and l_{i+1} (indices taken modulo 4). Find the least possible area of $Y_1Y_2Y_3Y_4$.

Proposed by: Derek Zhu

Answer: $\boxed{12675/44}$



WLOG let ABC be a triangle with $AB = 13, BC = 14, CA = 15$, and let X be any point on the circumference of ABC. WLOG let X be on the smaller arc $\widehat{AC}$ and other cases are analogous. If R is the radius and O is the circumcenter, then the area of the quadrilateral can be written as the sum of cyclic quadrilaterals OAY_1B, OBY_2C, OCY_3X, and OXY_4A, which are all pairs of congruent right triangles, each with one leg length as the radius. Thus, the sum of these four quadrilaterals can be evaluated as $R^2(\tan(\angle ACB)+\tan(\angle BAC)+\tan(\angle XBC)+\tan(\angle XBA))$. Note that since $\tan(\angle ACB)$ and $\tan(\angle BAC)$ are fixed, it suffices to minimize $\tan(\angle XBC)+\tan(\angle XBA)$, which occurs when $\angle XBC = \angle XBA$ as tan is convex on $[0, \frac{\pi}{2})$. So whatever arc X is on, the quadrilateral area is minimized when X is on the midpoint of the arc. The overall minimum is achieved when X is on the midpoint of minor arc of AC, because $\angle ABC$ is the largest of the three, so $\tan(\angle ABC) - (\tan(\angle XBC)+\tan(\angle XBA))$ is the largest among all three analogous calculations. Some calculations show $R = \frac{65}{8}$, $\tan(\angle ACB) = \frac{4}{3}$, $\tan(\angle CAB) = \frac{56}{33}$, and $\tan(\angle ABC) = \frac{12}{5}$, and $\tan(\angle XBC) = \tan(\angle XBA) = \tan(\frac{\angle ABC}{2}) = \frac{2}{3}$. Thus, our answer is $\left(\frac{65}{8}\right)^2\left(\frac{4}{3}+\frac{56}{33}+\frac{2}{3}+\frac{2}{3}\right) = \frac{12675}{44}$.

18. Charlie has a fair n-sided die (with each face showing a positive integer between 1 and n inclusive) and a list of n consecutive positive integer(s). He first rolls the die and if the number showing on top is k, he then uniformly and randomly takes a k-element subset from his list and calculates the sum of the numbers in his subset. Given that the expected value of this sum is 2020, compute the sum of all possible values of n.

Proposed by: Kyle Lee

Answer: $\boxed{78}$

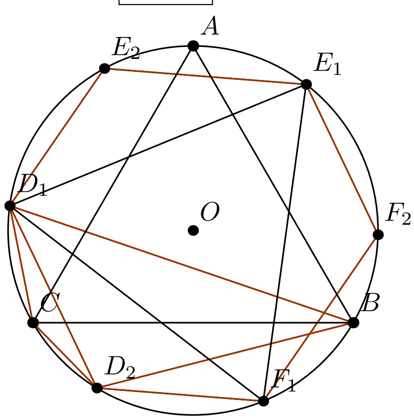
Let a be the starting number of the list and suppose $t = n-1$ so that the list has $t+1$ terms. The expected value of k is $\frac{1+(t+1)}{2} = \frac{t+2}{2}$, and the expected value of a term in the list is $\frac{a+(a+t)}{2} = \frac{2a+t}{2}$. Thus, the expected value of the sum of the k-element subset is $\frac{t+2}{2}\cdot\frac{2a+t}{2} = \frac{1}{4}(t+2)(2a+t)$.

Hence, we have $(t+2)(2a+t) = 8080 = 2^4\cdot 5\cdot 101$. Clearly $a > 1$ since 8080 is not a perfect square, so $2a+t > t+2$. Moreover, it is easy to see that both factors have the same parity and thus must be even. Then suppose $2a+t = 2p$ and $2+t = 2q$, where $p > q$ are factors of $2^2\cdot 5\cdot 101$, so that $t = 2q-2$. From here, it easy to see that the possible values of q are $1, 2, 4, 5, 10, 20$ (the corresponding values of t are $0, 2, 6, 8, 18, 38$). Thus, the sum of all possible values of n is $2(1+2+4+5+10+20) - 6(2) + 6 = 78$.

19. Let ABC be an equilateral triangle with unit side length and circumcircle Γ. Let D_1, D_2 be the points on Γ such that $BD_i = 3CD_i$. Let E_1, E_2 be the points on Γ such that $CE_i = 3AE_i$. Let F_1, F_2 be the points on Γ such that $AF_i = 3BF_i$. Then points $D_1, D_2, E_1, E_2, F_1, F_2$ are the vertices of a convex hexagon. What is the area of this hexagon?

Proposed by: Kevin Ren

Answer: $\boxed{\frac{181\sqrt{3}}{364}}$



Let $CD_1 = a$, $CD_2 = b$, and suppose $\angle BD_1C = 60°$, $\angle BD_2C = 120°$. By Law of Cosines on BD_1C, we get $a^2(1+9-3) = 1$, so $a = \frac{1}{\sqrt{7}}$. Similarly, $b^2(1+9+3) = 1$, so $b = \frac{1}{\sqrt{13}}$.

By Ptolemy's theorem on BD_1CD_2, we get $x = D_1D_2 = 6ab = \frac{6}{\sqrt{91}}$, so $x^2 = \frac{36}{91}$. We also observe $D_1E_1F_1$ is equilateral and also inscribed in Γ, so $D_1F_1 = BC = 1$.

Let $y = D_2F_1$. Then by Law of Cosines on $D_1D_2F_1$, we get $x^2+xy+y^2 = 1$, so $y = \frac{-x+\sqrt{4-3x^2}}{2}$. Thus, $xy = \frac{-x^2+\sqrt{x^2(4-3x^2)}}{2} = \frac{1}{2}\left(-\frac{36}{91}+\sqrt{\frac{36}{91}\cdot\frac{256}{91}}\right) = \frac{30}{91}$. By dissecting the hexagon into $D_1E_1F_1$ and three smaller triangles, we see that the area of the hexagon is $\frac{\sqrt{3}}{4}(1+3xy) = \frac{181\sqrt{3}}{364}$.

20. Let $\tau(n)$ be the number of positive divisors of n, let $f(n) = \sum_{d|n}\tau(d)$, and let $g(n) = \sum_{d|n} f(d)$. Let P_n be the product of the first n prime numbers, and let $M = P_1P_2\cdots P_{2021}$. Then $\sum_{d|M}\frac{1}{g(d)} = \frac{a}{b}$, where a, b are relatively prime positive integers. What is the remainder when $\tau(ab)$ is divided by 2017? (Here, $\sum_{d|n}$ means a sum over the positive divisors of n.)

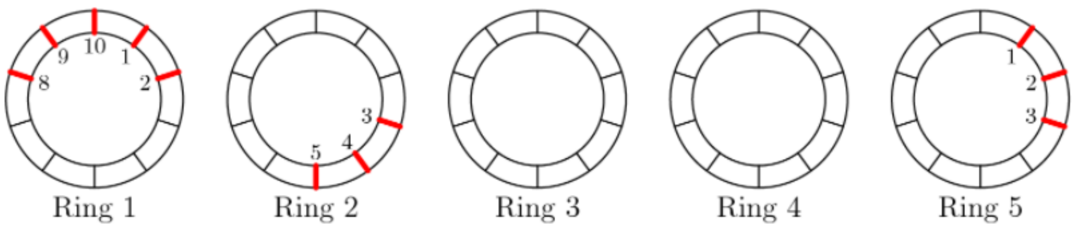
Proposed by: Kevin Ren

We compute $\tau(p^n) = n+1$, $f(p^n) = 1+2+\cdots+n+1 = \binom{n+2}{2}$, and (by Hockey Stick identity) $g(p^n) = \binom{n+3}{3}$. Thus,

$$\begin{aligned}\sum_{d|p^n} \frac{1}{g(d)} &= \sum_{k=0}^{n} \frac{6}{(k+1)(k+2)(k+3)} \\ &= 3\sum_{k=0}^{n}\left(\frac{1}{(k+1)(k+2)} - \frac{1}{(k+2)(k+3)}\right) \\ &= 3\left(\frac{1}{2} - \frac{1}{(n+2)(n+3)}\right) \\ &= \frac{3(n+1)(n+4)}{2(n+2)(n+3)}.\end{aligned}$$

Let the n-th prime be p_n. Then $M = p_1^{2021}p_2^{2020}\cdots p_{2021}$, so our answer is $\prod_{n=1}^{2021}\frac{3(n+1)(n+4)}{2(n+2)(n+3)} = \left(\frac{3}{2}\right)^{2021}\cdot\frac{2}{2023}\cdot\frac{2025}{4} = \frac{3^{2021}\cdot 2025}{2^{2022}\cdot 2023} = \frac{3^{2025}\cdot 5^2}{2^{2022}\cdot 7\cdot 17^2}$. Thus, $\tau(ab) = 2026\cdot 2023\cdot 3^2\cdot 2$, so the remainder upon division by 2017 is $9\cdot 6\cdot 9\cdot 2 = \boxed{972}$.

21. Sarah has five rings (numbered 1 through 5), each with ten rungs labeled 1 through 10. Rung i is adjacent to rung $i+1$ for $1 \le i \le 9$, and rung 10 is adjacent to rung 1. How many ways can Sarah paint some (possibly none) of the rungs red such that in each ring, the red rungs form a contiguous block, and the total number of red rungs across the five rings is divisible by 11? (For example, Sarah can paint rungs $8, 9, 10, 1, 2$ on ring 1, rungs $3, 4, 5$ on ring 2, no rungs on rings 3 and 4, and rungs $1, 2, 3$ on ring 5.)



Proposed by: Kevin Ren, Derek Zhu, Kyle Lee, and Freya Edholm

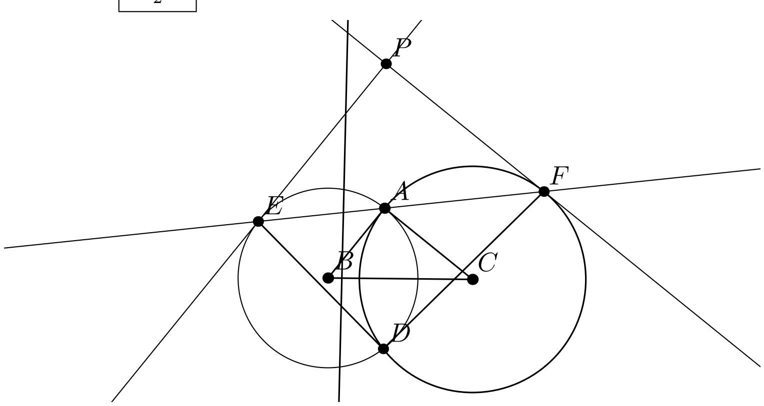
Answer: $\boxed{\frac{92^5+21\cdot 9^5}{11}}$

For a given ring, there is 1 way to color zero rungs red, 1 way to color all ten rungs red, and 10 ways to color k rungs red, for $1 \le k \le 9$. Thus, our answer is the sum of the x^{11a} coefficients in $(1+10x+10x^2+\cdots+10x^9+x^{10})^5$. This is done by taking the roots of unity filter, for ω a primitive 11-th root of unity: $\frac{1}{11}\sum_{i=0}^{10}(1+10\omega^i+10\omega^{2i}+\cdots+10\omega^{9i}+\omega^{10i})^5 = \frac{1}{11}(92^5 - 9^5\sum_{i=1}^{10}(1+\omega^i)^5$. By roots of unity filter in reverse, we know $\sum_{i=0}^{10}(1+\omega^i)^5$ is 11 times the sum of the $x^0, x^{11}, \cdots$ coefficients of $(1+x)^5$, which yields $11\cdot 1 = 11$. Hence $\sum_{i=1}^{11}(1+\omega^i)^5 = 11 - 32 = -21$ and the answer is $\frac{92^5+21\cdot 9^5}{11}$.

22. Let ABC be a triangle with $AB = 20$, $AC = 21$, and $\angle BAC = 90°$. Suppose Γ_1 is the unique circle centered at B and passing through A, and Γ_2 is the unique circle centered at C and passing through A. Points E and F are selected on Γ_1 and Γ_2, respectively, such that E, A, F are collinear in that order. The tangent to Γ_1 at E and the tangent to Γ_2 at F intersect at P. Given that $PA \perp BC$, compute the area of PBC.

Proposed by: Kyle Lee and Kevin Ren

Answer: $\boxed{\frac{1261}{2}}$



Since $PA \perp BC$, we have P lies on the radical axis of Γ_1 and Γ_2, so $EP = FP$. If D is the reflection of A over BC, then by tangency, we have $\angle PEF = \angle EDA$ and $\angle PFE = \angle FDA$. Thus, since PEF is isosceles, we have $\angle PEF = \angle PFE$, and so $\angle EDA = \angle FDA$. Now by a well-known spiral similarity lemma, we have $DEF \sim DBC$, so $\angle EDF = \angle BDC =$

$\angle BAC = 90°$. Since $\angle EDF = \angle EDA + \angle FDA$, we get $\angle EDA = \angle FDA = 45°$, so $\angle EBA = 2\angle EDA = 90°$ and $\angle ACF = 2\angle FDA = 90°$. Thus, EBA and ACF are $45-45-90$ right triangles, so $EA = 20\sqrt{2}, FA = 21\sqrt{2}$. Furthermore, $\angle PEF = \angle EDA = 45°$ and $\angle PFE = \angle FDA = 45°$, so PEF is also a $45-45-90$ right triangle. If we let K be the midpoint of EF, then $KA = \frac{1}{2}$ and $KP = \frac{EF}{2} = \frac{41}{2}$, so by Pythagorean theorem, we get $PA = \sqrt{(\frac{1}{2})^2 + (\frac{41}{2})^2} = 29$. Another Pythagorean theorem on ABC tells us $BC = 29$. Finally, the area of PBC is the sum of the areas of ABC and $PBAC$, which equals $\frac{20\cdot 21 + 29\cdot 29}{2} = \frac{1261}{2}$.

23. Given real numbers x, y, z, w such that $(x+y+2z)(x+z+3w) = 1$, what is the minimum possible value of $x^2+y^2+z^2+w^2$?

Proposed by: Kevin Ren

Answer: $\boxed{\frac{2\sqrt{66}-6}{57}}$

The key idea is to apply AM-GM and Cauchy-Schwarz in a clever way.

Lemma: Let $\vec{a}, \vec{b}, \vec{v}$ be vectors. $2(\vec{v}\cdot\vec{a})(\vec{v}\cdot\vec{b}) \le (|\vec{a}||\vec{b}| + \vec{a}\cdot\vec{b})|\vec{v}|^2$

Proof. Without loss of generality, let $|\vec{a}| = |\vec{b}| = 1$. By AM-GM and Cauchy-Schwarz, we get

$$4(\vec{v}\cdot\vec{a})(\vec{v}\cdot\vec{b}) \le (\vec{v}\cdot(\vec{a}+\vec{b}))^2 \le |\vec{v}|^2|\vec{a}+\vec{b}|^2 = |\vec{v}|^2(|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}) = |\vec{v}|^2(2 + 2\vec{a}\cdot\vec{b}).$$

Dividing by 2 gives the desired result. □

Using Lemma, we get $2 = 2(x+y+2z)(x+z+3w) \le (\sqrt{66}+3)(x^2+y^2+z^2+w^2)$, so the answer is $\frac{2}{\sqrt{66}+3} = \frac{2\sqrt{66}-6}{57}$.

24. The center cell of a 5×5 square grid is removed. Determine the number of ways to color the remaining 24 cells one of four colors (cyan, magenta, yellow, and black) such that any 2×2 square of cells not containing the center cell contains cells of all four colors.

Proposed by: Ankan Bhattacharya

Answer: $\boxed{4608}$

Solution by Ben Qi.

Let $c_{i,j}$ denote the color at cell (i,j) for all $0 \le i \le 4, 0 \le j \le 4$ (aside from $(i,j) = (2,2)$, which is missing).

We will color the cells in the following order:

(a) First, the cells with $0 \le i \le 1, 0 \le j \le 1$.

(b) Second, the cells with $3 \le i \le 4, 3 \le j \le 4$.

(c) Third, the cells with $3 \le i \le 4, 0 \le j \le 1$ or $0 \le i \le 1, 3 \le j \le 4$.

(d) Finally, the cells with $i = 2$ or $j = 2$.

Each step in the above order contributes some multiplicative factor, and the final answer will be equal to the product of all these factors.

To start off, the first step contributes a factor of $4! = 24$.

We claim that the third step always contributes a factor of either 0 or 1 depending on the colors in the second step; i.e., fixing the colors of the bottom-left and top-right 2×2 squares also fixes the colors of the bottom-right and top-left 2×2 squares as well (assuming it is possible to complete the coloring).

In order to satisfy the given condition for the bottom-right portion of the grid ($2 \le i \le 4, 0 \le j \le 2$), we must have all of the following set equalities:

$$\{c_{0,0}, c_{0,1}\} = \{c_{2,0}, c_{2,1}\} = \{c_{4,0}, c_{4,1}\}$$
$$\{c_{1,0}, c_{1,1}\} = \{c_{1,0}, c_{3,1}\}$$
$$\{c_{3,0}, c_{4,0}\} = \{c_{3,2}, c_{4,2}\} = \{c_{3,4}, c_{4,4}\}$$
$$\{c_{3,1}, c_{4,1}\} = \{c_{3,3}, c_{4,3}\}$$

This implies that if we know the positions of the magenta cells in both the bottom-left and top-right 2×2 squares (for example, say $(i_0, j_0) = (0,1)$ and $(i_1, j_1) = (3,4)$ and $c_{i_0,j_0} = c_{i_1,j_1} =$ (magenta)), then we may determine the position of the magenta cell in the bottom-right square; i.e., $c_{4-i_0,4-j_1} =$ (magenta) (in this example, $(4-i_0, 4-j_1) = (4,0)$). Similar reasoning determines the position of the magenta cell in the top-left 2×2 square; i.e., $c_{4-i_1,4-j_0} =$ (magenta).

So both a necessary and set of conditions that the colors of the top-right 2×2 squares must satisfy is as follows:

- $c_{3,3}, c_{4,3}, c_{3,4}, c_{4,4}$ are all distinct.
- $\{c_{0,0}, c_{0,1}\} \ne \{c_{3,3}, c_{4,3}\}$ and $\{c_{0,0}, c_{0,1}\} \ne \{c_{3,4}, c_{4,4}\}$ (so it is possible to color the bottom-right 2×2 square).
- $\{c_{0,0}, c_{1,0}\} \ne \{c_{3,3}, c_{3,4}\}$ and $\{c_{0,0}, c_{1,0}\} \ne \{c_{4,3}, c_{4,4}\}$ (so it is possible to color the top-left 2×2 square).

To count the number of ways to color the top-right 2×2 square after the bottom-left 2×2 square has already been colored such that all of these conditions are satisfied, first, choose the ordered pair (i,j) satisfying $3 \le i, j \le 4$ such that $c_{i,j} = c_{0,0}$ (contributing a factor of four). Then there are always three ways to color in the remaining three cells of the top-right 2×2 square such that $\{c_{0,0}, c_{0,1}\} \ne \{c_{i,j}, c_{7-i,j}\}$ and $\{c_{0,0}, c_{1,0}\} \ne \{c_{i,j}, c_{i,7-j}\}$. So the second and third steps combined contribute a factor of $4\cdot 3 = 12$.

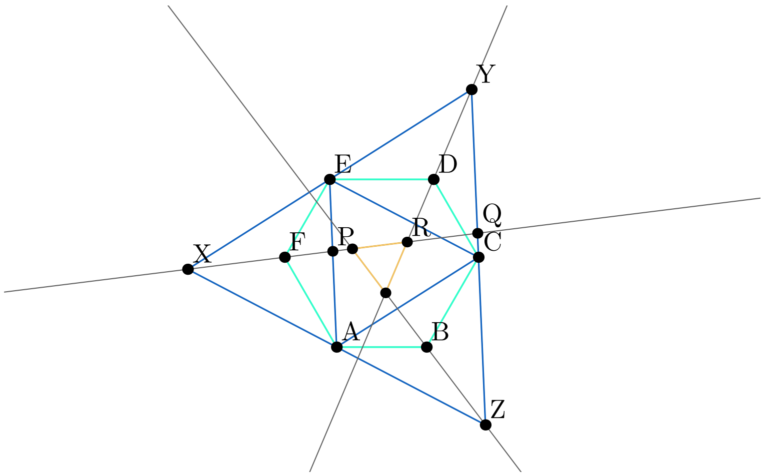
Finally, the fourth step contributes a factor of $2^4 = 16$, as we know the values of each of the following sets of colors: $\{c_{2,0}, c_{2,1}\}$, $\{c_{2,3}, c_{2,4}\}$, $\{c_{0,2}, c_{1,2}\}$, $\{c_{3,2}, c_{4,2}\}$ and each set contributes a factor of two.

So our answer is $24\cdot 12\cdot 16 = 4608$.

25. Convex equiangular hexagon $ABCDEF$ has $AB = CD = EF = \sqrt{3}$ and $BC = DE = FA = 2$. Points X, Y, and Z are situated outside the hexagon such that AEX, ECY, and CAZ are all equilateral triangles. Compute the area of the region bounded by lines XF, YD, and ZB.

Proposed by: Kyle Lee

Answer: $\boxed{\frac{7\sqrt{3}+6}{37}}$



Note that by symmetry the region is a smaller equilateral triangle. We will compute its area by taking the area of the largest equilateral triangle $\triangle XYZ$ and subtract off three identical triangular regions.

WLOG consider the triangular region with longest side XY. Suppose $P = AE \cap XF$. Then since $ABCDEF$ is cyclic and this circle is in fact the inscribed circle of $\triangle XYZ$, we have that line FP is a symmedian of $\triangle AFE$.

Then if we let $Q = XF \cap YZ$, we have $\frac{ZQ}{QY} = \frac{AP}{PE} = (\frac{2}{\sqrt{3}})^2 = \frac{4}{3}$. Hence, $QY = \frac{6s}{7}$, where the side length of $\triangle XYZ$ is just

$$2s = 2\sqrt{2^2 + (\sqrt{3})^2 - 2 \cdot 2 \cdot \sqrt{3} \cdot \cos(120)} = 2\sqrt{7 + 2\sqrt{3}}.$$

Suppose $R = XF \cap YD$. By the ratio lemma, we have $\frac{RQ}{RX} = \frac{6/7}{2} \cdot \frac{3}{4} = \frac{9}{28}$. Lastly, we have by area ratios that $\frac{[YQR]}{[YRX]} = \frac{RQ}{RX} = \frac{9}{28}$, so

$$[YRX] = \frac{1}{2}(\frac{28}{28+9}) \cdot \frac{6}{7} \cdot 2 \cdot s^2 \cdot \sin(60) = \frac{12s^2\sqrt{3}}{37}.$$

The desired answer is then

$$\frac{\sqrt{3}}{4}(2s)^2 - 3 \cdot \frac{12s^2\sqrt{3}}{37} = \frac{s^2\sqrt{3}}{37} = \frac{6+7\sqrt{3}}{37}.$$

Remark. Once we have $\frac{ZQ}{QY} = \frac{4}{3}$, we can simply finish with Routh's Theorem. `https://brilliant.org/wiki/rouths-theorem/`

26. How many pairs of integers (a, b) satisfy $1 \le a < 1001^3$, $1 \le b < 1001^2$, and $1001^3 \mid a^3 + ab$?

Proposed by: Jeffery Yu and Kevin Ren

Answer: $\boxed{48271224}$

If we make it $0 \le a < N^3$ and $0 \le b < N^2$, then the count for $N = 1001$ is the product of the counts for $N = 7, 11, 13$ (thanks to the Chinese Remainder theorem). For $p = 7, 11, 13$, we look at the following cases:

- If $\gcd(a, p) = 1$, then $p^3 \mid a^2 + b$. Thus, $-b$ must be a quadratic residue mod p^3, so it must be a quadratic residue mod p. There are $\frac{p(p-1)}{2}$ such residues between 0 and $p^2 - 1$ that are coprime to p, and each value of b gives 2 values for a, so we get $p(p-1)$ choices for (a, b).
- If $p \mid a$, then we want $p^3 \mid ab$. If $v_p(a) = 1$, then there are $p(p-1)$ choices for a and 1 choice for b. If $v_p(a) = 2$, then there are $p - 1$ choices for a and p choices for b. If $v_p(a) = 3$, then there is 1 choice for a and p^2 choices for b.

Adding gives $p(p-1) + p(p-1) + p(p-1) + p^2 = p(4p-3)$. Multiplying this number for $p = 7, 11, 13$ gives a preliminary answer of $1001 \cdot 25 \cdot 41 \cdot 49$.

Now we subtract $a = 0$ and $b = 0$. If $a = 0$ then b has 1001^2 choices. If $b = 0$ then $1001 \mid a$, so a has 1001^2 choices. Finally, there is one ordered pair with $a = 0$ and $b = 0$. Hence, by PIE, the answer is

$$1001 \cdot (25 \cdot 41 \cdot 49 - 1001 - 1001) + 1 = 48271224.$$

27. You are participating in a virtual stock market, with many different stocks. For a stock S, there is a list of prices where the ith number is the price of the stock on day i. On each day i, you are given the stock's current price (in dollars), and you can either buy a share of stock S, sell your share of stock S, or do nothing, but you may only take one of these actions per day, and you may not have more than one share of stock S at a time. Each stock is independent, so for example on the first day, you may buy a share of S and a share of T, and on the second day you may sell your share of T.

At USMCA Trading LLC, you are given 2021! different stocks, where each stock's list of prices corresponds to a unique permutation of the first 2021 positive integers, to trade for 2021 days. You start out with M dollars, and at the end of 2021 days, you end up with N dollars. Assume M is large enough so that you can never run out of money during the 2021 days. What is the maximum possible value of $N - M$?

Proposed by: Derek Zhu

Answer: $\boxed{\frac{2020 \cdot 2022!}{6}}$

Let $n = 2021$. The strategy is that whenever the price of a stock on the next day is higher than that of today, you should buy the stock today if you currently don't have it and do nothing if you currently have it. Otherwise, you should sell the stock today if you have it and do nothing if you currently don't have it. This means for all $n!$ permutations π of $(1, \cdots, n)$, between each consecutive days i and $(i+1)$, we are expected to make a profit of 0 or $\pi(i+1) - \pi(i)$.

We need to evaluate $\sum_\pi \sum_{i=1}^{n-1}(\max(0, \pi(i+1) - \pi(i)))$ across all permutations π. For all pairs of positive integers (a, b) where $1 \le a < b \le n$, there are $(n-1)!$ permutations that have prices on two consecutive days be a and b, because there are $(n-1)$ ways to place the pair (a, b) and $(n-2)!$ ways to arrange the rest of the prices. Between each of these consecutive days, we make a profit of $(b - a)$ dollars, so the expression becomes $\sum_{1 \le a < b \le n}(n-1)! \cdot (b - a)$, or $(n-1)! \cdot \sum_{i=1}^{n-1}(n-i) \cdot i$ because there are $(n - i)$ pairs (a, b) such that $(b - a) = i$. Then $\sum_{i=1}^{n-1}(n-i) \cdot i = n \cdot \sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} i^2 = n \cdot \frac{n(n-1)}{2} - \frac{(n-1)n(2n-1)}{6} = \frac{n(n-1)(n+1)}{6}$.

Plugging in $n = 2021$, our final answer is $(n-1)! \cdot \frac{(n-1)n(n+1)}{6}$, or $2020! \cdot \binom{2022}{3}$.

28. How many functions $f : \mathbb{Z} \to \{0, 1, 2, \cdots, 2020\}$ are there such that $f(n) = f(n+2021)$ and $2021 \mid f(2n) - f(n) - f(n-1)$ for all integers n?

Proposed by: Kevin Ren

Answer: $\boxed{2021^{11}}$

From $n = -1$ we get $f(-1) = 0$. Define $g(n) = f(n) - f(n-2) \bmod 2021$; then we see that $g(2n) = g(n)$ for all n. We may partition the integers mod 2021 into equivalence classes $C_1, C_2, \cdots, C_k$, where a, b belong to the same class if $b \equiv 2^i a \pmod{2021}$ for some integer i. For a class C_i, we may define $g(C_i) = g(x)$ for any $x \in C_i$; this is well-defined since $g(2n) = g(n)$ for all n. The condition $\sum_{n=0}^{2020} g(n) = 0 \bmod 2021$ is equivalent to

$$\sum_{i=1}^{k} |C_i| g(C_i) = 0 \pmod{2021},$$

and this is the only condition on g: any g satisfying this condition gives rise to a unique f that satisfies the problem conditions. Let C_1 be the equivalence class containing 0: then clearly 0 is the only element of C_1. Thus,

$$g(0) = -\sum_{i=2}^{k} |C_i| g(C_i) \mod 2021,$$

so $g(0)$ is uniquely determined from the other $g(C_i)$. There are 2021 choices for each $g(C_i)$ mod 2021, so our answer is 2021^{k-1}. Thus, it suffices to compute k. To do this, we need to know something about the classes. Each class is of the form $\{x, 2x, \ldots, 2^{d-1}x\}$ mod 2021, so the size of the class is the smallest d such that $2^d x \equiv x \pmod{2021}$. To find the smallest such d, we compute the order of 2 mod 43 and mod 47.

Since $47 \equiv -1 \pmod 8$, we get by quadratic reciprocity (or just direct computation) that $2^{23} \equiv 1 \pmod{47}$. Since 23 is prime and $2^1 \not\equiv 1 \pmod{47}$, we find the order of 2 mod 47 is 23.

We also compute $2^{14} \equiv 1 \pmod{43}$, while $2^7, 2^2, 2^1$ are not 1 mod 43, so the order of 2 mod 43 is 14.

Thus, given an element x, we can compute the size of the class containing x. We will now count the number of classes via casework on x:

Case 1. x is coprime to 2021. The least d with $2021 \mid 2^d - 1$ is $23 \cdot 14$, so the class of x has $23 \cdot 14$ elements. There are $\phi(2021) = 42 \cdot 46$ residues mod 2021 that are coprime to 2021, so there are $\frac{42 \cdot 46}{23 \cdot 14} = 6$ classes involving elements coprime to 2021.

Case 2. $43 \mid x$ but $47 \nmid x$. The least d with $2021 \mid 43(2^d - 1)$ is 23, so the class of x has 23 elements. There are 46 residues mod 2021 that are divisible by 43, so there are $\frac{46}{23} = 2$ classes involving nonzero elements divisible by 43.

Case 3. $47 \mid x$ but $43 \nmid x$. The least d with $2021 \mid 47(2^d - 1)$ is 14, so the class of x has 14 elements. There are 42 residues mod 2021 that are divisible by 47, so there are $\frac{42}{14} = 3$ classes involving nonzero elements divisible by 47.

Case 4. $x = 0$. The class of 0 is $\{0\}$, and there is one such class.

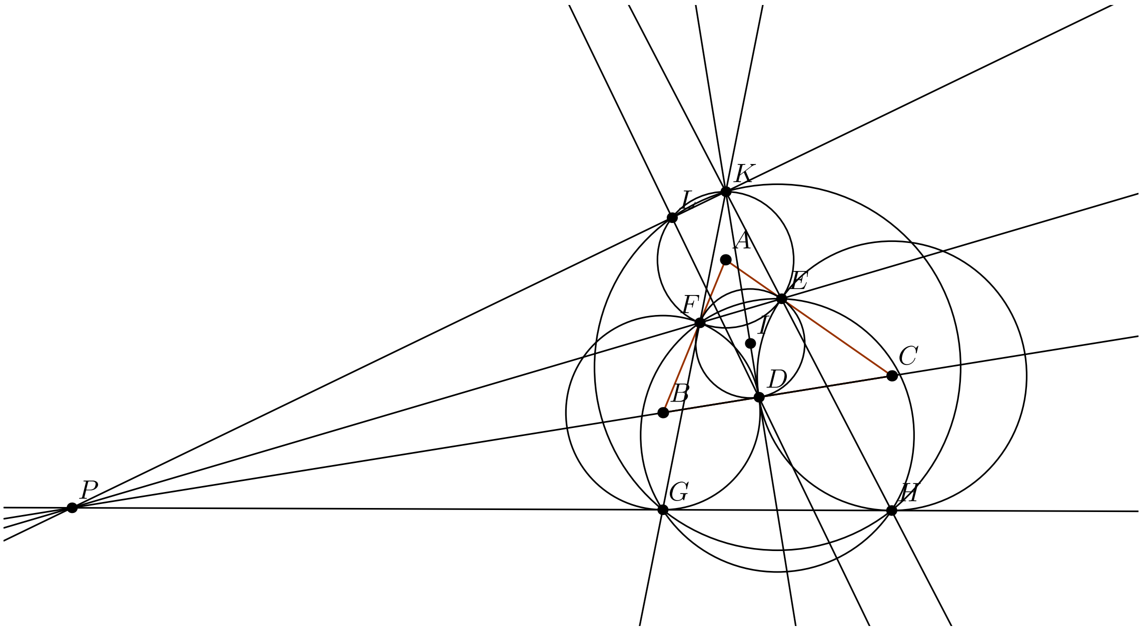
As a result, we get $k = 6 + 2 + 3 + 1 = 12$, so the answer is $2021^{k-1} = 2021^{11}$.

29. Three circles $\Gamma_A, \Gamma_B, \Gamma_C$ are externally tangent. The circles are centered at A, B, C and have radii $4, 5, 6$ respectively. Circles Γ_B and Γ_C meet at D, circles Γ_C and Γ_A meet at E, and circles Γ_A and Γ_B meet at F. Let GH be a common

external tangent of Γ_B and Γ_C on the opposite side of BC as EF, with G on Γ_B and H on Γ_C. Lines FG and EH meet at K. Point L is on Γ_A such that $\angle DLK = 90^\circ$. Compute $\frac{LG}{LH}$.

Proposed by: Kevin Ren

Answer: $\boxed{\frac{5\sqrt{10}}{18}}$



First, a simple fact.

Lemma. $\angle GDH = 90^\circ$.

Proof. Let M be the midpoint of GH. Then by equal tangents, $MG = MD$ and $MD = MH$, hence M is the center of the circumcircle of GDH and $\angle GDH = 90^\circ$. ☐

Let EF and BC meet at P. A lemma in projective geometry tells us that $(B, C; P, D) = -1$. Since Γ_B and Γ_C are externally tangent at D, we see that P is the external center of homothety of Γ_B and Γ_C. Thus, GH also passes through P.

Using Lemma, we see that $\angle HDG = 90^\circ - \angle CDH = 90^\circ - \angle CHD = \angle DHG$. Thus, the circumcircle of DGH is tangent to PD. By Power of a Point, we thus get $PF \cdot PE = PD^2 = PG \cdot PH$, so $EFGH$ is cyclic. Thus, FG and EH intersect on the radical axis of Γ_B and Γ_C, and so $KD \perp BC$.

Next, we claim that K lies on Γ_A. Indeed, if FG meets Γ_A at K_1, then by looking at the homothety sending Γ_B to Γ_A, we see that the tangent to Γ_A at K_1 is parallel to the tangent to Γ_B at F, which is GH. Similarly, if EH intersects Γ_A at K_2, then the tangent to Γ_A at K_2 is also parallel to GH. This means the tangents to Γ_1 at K_1 and K_2 are actually parallel, so $K_1 = K_2$. Thus, K lies on Γ_A, as claimed.

Now, the condition $\angle DLK = 90^\circ$ (combined with $DK \perp BC$) implies that the circumcircle of DLK is tangent to BC at D. Hence, we apply the radical axis theorem on Γ_A, the incircle of ABC, and the circumcircle of DLK to show that LK also passes through P. Now, Power of a Point tells us $PL \cdot PK = PE \cdot PF = PG \cdot PH$, hence $GHKL$ is cyclic. Hence, L is the Miquel point of $EFGH$. As a result, we have $LFG \sim LEH$, so $\frac{LG}{LH} = \frac{FG}{EH}$. We present two possible finishes.

Solution 1. Since $PFG \sim PHE$, we have $\frac{LG^2}{LH^2} = \frac{FG^2}{EH^2} = \frac{PF \cdot PG}{PE \cdot PH}$.

We have $\frac{PG}{PH} = \frac{r_B}{r_C} = \frac{5}{6}$, and by Menelaus on AEF, we get $\frac{PF}{PE} = \frac{BF}{BA} \cdot \frac{AC}{EC} = \frac{r_B}{r_A + r_B} \cdot \frac{r_A + r_C}{r_C} = \frac{5}{6} \cdot \frac{10}{9}$. Thus, $\frac{MF}{ME} = \frac{FG}{EH} = \frac{5}{6}\sqrt{\frac{10}{9}} = \frac{5\sqrt{10}}{18}$.

Solution 2. (kevinmathz and Awesome_guy) Power of a Point gives $KF \cdot KG = KE \cdot KH$. Note that Γ_A and Γ_B are homothetic, so $KF = 4a$ and $FG = 5a$ for some a. Also, Γ_A and Γ_C are homothetic, so $KE = 2b$ and $EH = 3b$ for some b. Thus, our PoP equation gives $4a \cdot 9a = 2b \cdot 5b$, so $18a^2 = 5b^2$. Our answer is $\frac{FG}{EH} = \frac{5a}{3b} = \frac{5\sqrt{10}}{18}$.

30. I start with a sequence of letters $A_1A_2 \cdots A_{2021}A_1A_2 \cdots A_{2021}A_1A_2 \cdots A_{2021}$. I go through $i = 1, 2, 3, \cdots, 6062$ in order, and for each i, I can choose to swap letters i and $i+1$. Let N be the number of distinct strings I can end up with. What is the remainder when N is divided by 2017?

Proposed by: Freya Edholm, Derek Zhu, Kyle Lee, and Kevin Ren

Answer: $\boxed{2014}$

This is a tricky Principle of Inclusion-Exclusion problem. Let $n = 2021$. For ease of reference, let the string be $A_1A_2 \cdots A_nB_1B_2 \cdots B_nC_1C_2 \cdots C_n$, except we agree that $A_i = B_i = C_i$.

Without restrictions, there are 2^{3n-1} possible strings: one for each possible swap. However, we overcount the possibilities when we swap some A_i with B_i (or B_i with C_i) at some point in the algorithm. Thus, we subtract one from our count for each such possibility.

(For example, if $n = 2$, then if we swap the first and second letters, resulting in $A_2A_1B_1B_2C_1C_2$, then swapping the second and third letters doesn't affect the string.)

Note that there is at most one i such that A_i can swap with B_i, and there is at most one j such that B_j can swap with C_j. Now, we are ready to do PIE.

Suppose A_i can swap with B_i. If $i = 1$, then we have n forced swaps (A_k with A_{k+1} for $1 \le k \le n-1$, and A_1 with B_1). Thus, we have $2n-1$ possible swaps remaining, and so we have 2^{2n-1} possibilities for the remaining swaps. If $i \ne 1$, then we have $n+1$ forced swaps (A_{i-1} does not swap with A_i, and we swap letters k and $k+1$ for $i \le k \le n+i-1$). Thus, we have $2n-2$ possible swaps remaining, and so we have 2^{2n-2} possibilities for the remaining swaps. A similar argument holds if B_j can swap with C_j, except we always have $n+1$ forced swaps and 2^{2n-2} possibilities for the remaining swaps. Thus, our overcount is

$$2^{2n-1} + (2n-1)2^{2n-2} = (2n+1)2^{2n-2}.$$

Finally, we need add back possibilities where A_i swaps with B_i and B_j swaps with C_j. Then we must have $1 \le i < j \le n$. If $i = 1$, then we have $n + (n+1)$ forced swaps, and there are $n-1$ choices for j, so we have $(n-1)2^{n-2}$ choices for the remaining swaps. If $i \ne 1$, then we have $(n+1) + (n+1)$ forced swaps, and there are $\binom{n-1}{2}$ choices for i, j, so we have $\binom{n-1}{2}2^{n-3}$ choices for the remaining swaps. Thus, we need to add back

$$(n-1)2^{n-2} + \binom{n-1}{2}2^{n-3} = (n-1)(n+2)2^{n-4}$$

possibilities.

Thus, by PIE, we can compute $N = 2^{3n-1} - (2n+1)\cdot 2^{2n-2} + (n+2)(n-1)\cdot 2^{n-4}$. We have by Fermat's little theorem $2^{n-5} \equiv 1 \pmod{n-4}$, so

$$N \equiv 2^{14} - 9\cdot 2^8 + 18\cdot 2 \equiv 2014 \pmod{2017}.$$

(Remark: N is also the number of compositions of $3n$ with no part greater than n. `https://oeis.org/search?q=13%2C149%2C1490&sort=&language=english&go=Search`)